THE MADISON COLLOQUIUM.

THE Seventh Colloquium of the American Mathematical Society was held in connection with its twentieth summer meeting at the University of Wisconsin, Madison, Wis. At the April meeting of 1911 the Council appointed a committee consisting of Professors Van Vleck, Moore, Osgood, and the Secretary to make the arrangements for the meeting and colloquium. The courses of lectures were announced in the preliminary circular of May, 1913, and printed syllabi were issued in advance of the meeting. The colloquium opened on Wednesday morning, September 10, and extended through the rest of the week. The following fifty-one persons were in attendance, a larger number than at any previous colloquium:

Professor R. C. Archibald, Professor R. P. Baker, Professor G. N. Bauer, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Professor G. A. Bliss, Professor Oskar Bolza, Professor W. H. Bussey, Dr. G. R. Clements, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Mr. H. J. Ettlinger, Professor W. W. Hart, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Dr. Dunham Jackson, Dr. A. J. Kempner, Mr. Barnem Libby, Professor G. H. Ling, Professor A. C. Lunn, Professor H. W. March, Professor Max Mason, Mr. J. S. Mikesh, Professor C. N. Moore, Professor E. H. Moore, Professor W. F. Osgood, Professor R. G. D. Richardson, Professor W. J. Risley, Professor W. H. Roever, Miss Lulu Runge, Dr. Mildred Sanderson, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. T. M. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor E. R. Smith, Professor A. L. Underhill, Dr. S. E. Urner, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor R. E. Wilson, Professor H. C. Wolff, Professor B. F. Yanney, Professor Alexander Ziwet.

Two courses of five lectures each were given:

I. Professor L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry."

II. Professor W. F. OSGOOD: "Selected topics in the theory of analytic functions of several complex variables."

Abstracts of the lectures follow below. The lectures will soon be published in full by the Society as Volume IV of the Colloquium Series.

I. In the first lecture, Professor Dickson discussed the invariants of quadratic forms

$$q_m \equiv \beta_{11}x_1^2 + 2\beta_{12}x_1x_2 + \cdots + \beta_{mm}x_m^2, \qquad \beta_{ij} = \beta_{ji},$$

$$D \equiv |\beta_{ij}| \neq 0,$$

for the case in which the β_{ij} are complex numbers (the algebraic case), and also for the case in which they are integers, considered as equivalent if congruent modulo p (the number-theory case); forms of the latter type are called modular forms.

The invariants are obtained in both cases by a general method, based upon the classification of the forms in such a way that any two forms of the same class shall be convertible into each other by means of a transformation

$$x_j = \sum_{i=1}^{n} \alpha_{ji}X_i,$$

where in the algebraic case the α_{ji} are complex numbers and $|\alpha_{ji}| = 1$, and in the number-theory case they are integers, reducible modulo p, and $|\alpha_{ji}| \equiv 1 \pmod{p}$. It is then clear that any function $\varphi(\beta_{11}, \cdots, \beta_{mm})$ which takes the same value for all the forms of any one class must be an invariant of q_m. It was shown that in the algebraic case the value D of the determinant $|\beta_{ij}|$ and the rank r of this determinant form a fundamental system of invariants, i. e., that every invariant $\varphi(\beta_{11}, \cdots, \beta_{mm})$ is a single-valued function of D and r; furthermore, every rational integral invariant was shown to be a polynomial in D. In the number-theory case, it was found that, if the principal minors of order r of the determinant $|\beta_{ij}|$ are designated by $M_1, \cdots, M_n$, then a fundamental system of rational integral invariants is given by

$$D, I_0, A_1, \cdots, A_{m-1},$$

where

$$A_r = \Big[M_1^{\frac{p-1}{2}} + \sum_{i=2}^{n} M_i^{\frac{p-1}{2}}(1 - M_1^{p-1}) \cdots (1 - M_{i-1}^{p-1})\Big] \Pi(1 - d^{p-1}),$$

and

$$I_0 = \prod_{i,j=1}^{m} (1 - \beta_{ij}^{p^d - 1}),$$

d ranging over all the principal minors of $|\beta_{ij}|$ of order greater than r. An outline was given of a general theory of modular invariants of a system of forms.

Lecture II was devoted to an application of the methods developed in the first lecture to a study of the algebraic and the modular seminvariants of a binary form, i. e., of such functions of the coefficients a_i of a binary form

$$f \equiv \sum_{i=0}^{n} C_{n,i} a_i x^{n-i} y^i$$

as are invariant under the transformation $x = x' + ty'$, $y = y'$. The work was carried through in detail for the case $n = 4$, leading to the well-known result that if we put $t = -a_1/a_0$ $(a_0 \neq 0)$, then the form f goes over into a new form, whose coefficients, multiplied by proper powers of a_0, together with certain combinations I and J of them, form a fundamental system of seminvariants for f. It was found that the algebraic seminvariants were not sufficient to characterize all the different classes of forms into which the binary quartic forms were grouped; the modular seminvariants were then constructed and thereby a complete characterization of these classes was secured.

In the third lecture, Professor Dickson took up a discussion of the invariants of the group of transformations

$$x' = ax + by, \quad y' = cx + dy,$$

a, b, c, d being integers, such that $ad - bc \equiv 1 \bmod p$.

The fundamental system of invariants for this case was found to be

$$L \equiv y \prod_{i=0}^{p-1} (x - iy) \equiv yx^p - xy^p$$

and

$$Q \equiv (yx^{p^2} - xy^{p^2}) \div L.$$

In the case of n variables, there are n fundamental invariants. The "form problem" for the case of two variables,

i. e., the problem of determining x and y when $L(x, y)$ and $Q(x, y)$ are given, was then briefly discussed. It was shown that the form problem is completely solved when we find two linearly independent solutions of the congruence

$$z^{p^2} - \kappa z^p + \lambda^{p-1} \equiv 0 \pmod{p},$$

where $\lambda \equiv L^{p-1}(x, y)$, and $\kappa \equiv Q(x, y)$.

The lecturer then passed on to a presentation of the work of Hurwitz on formal invariants and its connection with the modular invariants. Considering for the purpose of illustration the form

$$f \equiv a_0x^2 + a_1xy + a_2y^2,$$

where a_0, a_1, a_2 are arbitrary variables, and subjecting this form to a transformation $x = ax' + by'$, $y = cx' + dy'$, $ad - bc \not\equiv 0 \pmod{p}$, which brings f to the form $b_0x'^2 + b_1x'y' + b_2y'^2$, a formal invariant may be defined as a function $F(a_0, a_1, a_2)$ of the coefficients of such character that

$$F(a_0, a_1, a_2) \equiv F(b_0, b_1, b_2) \pmod{p},$$

identically in a_0, a_1, a_2.

Of especial interest and importance here is the general theorem, due to Miss Sanderson, that from any modular invariant a formal invariant may be constructed, which will reduce to the modular invariant for integral values of the coefficients. This theorem is of great value in the construction of covariants. The lecture closed with a discussion of formal seminvariants.

The last two lectures were concerned with modular geometries, i. e., geometries in which only points with integral coordinates are considered as real points and in which two points are considered as identical if their coordinates are congruent. This work belongs entirely in the theory of numbers; the terminology, borrowed from geometry, suggests material and mode of treatment, and leads to results new and important in the theory of numbers. Particular reference was made to the recent work in this field of Veblen and Bussey, and to the work of Coble, who obtained an entry into the theory of theta-functions by carrying over into modular geometry some of the notions of projective geometry.

Some of the results of work in this field as stated by Professor Dickson are: All the polars of a conic in the space of a modular geometry of n dimensions pass through a fixed point, if n is odd. This point is called the *apex* of the conic. All the lines through the apex meet the conic in only one point. The tangential equation of the conic is *linear*.

The fundamental system of covariants of the ternary quadratic form were found, and the case of m variables was briefly considered.

Professor Dickson discussed in the closing lecture a theory of plane cubic curves with a real inflexion point. The number-theory character of the work led him to discover a rational reduction of the general cubic to the following normal form:

$$x^2y + gy^3 + hy^2z + \delta z^3 = 0,$$

where the point (1, 0, 0) is the real point of inflexion the existence of which among the real points of the curve was preassumed, and in which the coefficients g and h are simply related to the ordinary invariants of the cubic. The discussion of the points of inflexion, as to their reality and their configuration was carried on for various cases and for modular geometries with different moduli, one case being found in which the cubic has 9 real points of inflexion.

II. Professor Osgood's first lecture gave a general survey of the field, treating the following topics: Analytic functions of several variables; the factorial function and analytic continuation; existence theorems; Weierstrass's theorem of factorization; Jacobi's problem of inversion and the abelian functions; periodic functions; theta functions with several arguments; the theta theorem.

The second lecture was occupied with a number of general theorems: Rational and algebraic functions; sufficient conditions that a function of several variables be analytic; sufficient conditions that a function be rational, algebraic; on the associated radii of convergence of a power series.

The third lecture was devoted to singular points and analytic continuation: Introduction; non-essential singularities; essential singularities; removable singularities; analytic continuation by means of Cauchy's integral formula; application to the distribution of singularities; Levi's memoir of 1910; lacunary spaces; the boundary of the domain of

definition of $f(x, y)$; representation of certain meromorphic functions as quotients.

The fourth lecture dealt with implicit functions under the following headings: Weierstrass's theorem of factorization; a tentative generalization of the foregoing theorem; algebroid configurations; single-valued functions on an algebroid configuration; connectivity and the Riemann manifold; parametric representation im Kleinen; solution of a system of analytic equations, Weierstrass's theorem; a general theorem; special cases of the foregoing; the inverse of an analytic transformation.

In the final lecture, Professor Osgood discussed the prime function, introduced by Klein in the theory of the algebraic functions and their integrals. That function is not a function on the given algebraic configuration, but depends on the homogeneous coordinates of an allied configuration. It is possible, however, to obtain a function on the given configuration which has a single zero and is completely analogous to the elliptic θ or σ function.

ARNOLD DRESDEN.

THE VIENNA MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-fifth convention of the association of German naturalists and physicians at Vienna, September 22–25, under the presidency of Professor K. Rohn.

The Germans fully recognize the importance of the social side of these annual meetings, and make ample provision for the pleasant entertainment of the guests. Although the program was a long one, frequent excursions were arranged and a reception or concert each evening.

The session of Thursday afternoon was devoted to the administrative affairs of the society. Reports of the status of the Encyclopedia, the Euler commission, and the International commission were read, and a briefer statement was submitted concerning the publication of various other works supported in whole or in part by the society.

The sessions of Tuesday were held jointly with those of the sections of physics, geodesy, and astronomy at which the papers on applications of mathematics were presented.

The Vereinigung has now 770 members. The next annual meeting will be held in Hannover under the presidency of Professor C. Runge, of the University of Göttingen.

The following papers were read before the society.

(1) F. MEYER (Königsberg): "Bericht über neuere, besonders durch Arbeiten von Gordan veranlasste Fortschritte der Invariantentheorie."

(2) E. MÜLLER (Vienna): "Eine Weiterbildung der Grassmannschen Ausdehnungslehre im Sinne der Invariantentheorie."

(3) G. KOHN (Vienna): "Zur Geometrie der Würfe."

(4) F. HOČEVAR (Graz): "Ueber den Zusammenhang zwischen den irreduziblen Teilern einer Form und einem linearen System von Nullstellen der Form."

(5) R. WEITZENBÖCK (Graz): "Invarianten bezüglich Untergruppen der allgemeinen projectiven Gruppe."

(6) E. BLASCHKE (Vienna): "Aenderungen der Sterbewahrscheinlichkeiten mit der Zeit."

(7) L. G. DU PASQUIER (Neuenburg): "Eine neue Anwendung der simultanen Differentialgleichungen in der mathematischen Theorie der Lebensversicherung."

(8) A. EINSTEIN (Zürich): "Zum Gravitationsproblem."

(9) W. v. DYCK (Munich): "Ueber die Kepler-Manuskripte der Wiener Hofbibliothek."

(10) A. KORN (Berlin): "Zur Frage der internationalen Vereinheitlichung wichtiger Begriffe und Bezeichnungen in der Potential- und Elastizitätstheorie."

(11) E. WAELSCH (Brünn): "Zu den Minkowskischen Grundgleichungen der Elektrodynamik."

(12) R. MEHMKE (Stuttgart): "Ueber die zahlmässige und graphische Lösung eines Systemes von unendlich vielen Gleichungen in einer unendlichen Anzahl Unbekannten."

(13) C. JUEL (Kopenhagen): "Ueber Elementarflächen."

(14) F. ENGEL (Giessen): "Lie's Invariantentheorie der Berührungstransformationen und ihre Verallgemeinerung."

(15) H. LIEBMANN (Munich): "Die Entwicklung der Lehre von den Berührungstransformationen."

(16) K. ZINDLER (Innsbruck): "Ueber geschlossene Raumkurven."

(17) H. TIETZE (Vienna): "Ueber ein-eindeutige stetige Abbildungen von Flächen auf sich selbst."

(18) H. HAHN (Czernowitz): "Ueber stetige Abbildungen."

(19) E. HECKE (Göttingen): "Modulfunktionen von zwei Variabeln und ihre Anwendungen auf algebraische Körper."

(20) L. v. SCHRUTKA (Brünn): "Zur additiven Zahlentheorie."

(21) P. KOEBE (Leipzig): "Wesen und Ziele der Kontinuitätsmethode."

(22) J. PLEMELJ (Czernowitz): "Ueber den Verzerrungssatz von P. Koebe."

(23) J. PLEMELJ (Czernowitz): "Ueber die Abhängigkeit der Lösungen linearer Differentialgleichungen von den akzessorischen Parametern."

(24) F. DINGELDEY (Darmstadt): "Ueber ein gewisses Integral und eine einfache Darstellung der Kugelfunktionen."

(25) J. RADON (Vienna): "Unabhängigkeit von Kurvenintegralen vom Wege beim Bestehen von Nebenbedingungen."

(26) W. GROSS (Vienna): "Zur Theorie der unbestimmten Differentialgleichungen."

(27) R. SUPPANTSCHITSCH (Vienna): "Ueber die Axiomatik der Methode der kleinsten Quadrate."

(28) E. HÄNTZSCHEL (Berlin): "Bedingungen für die Lösbarkeit eines Fermatschen Problems."

(29) O. PERRON (Tübingen): "Ueber eine eigentümliche Schwierigkeit bei der Integration gewisser scheinbar sehr einfacher Differentialgleichungen."

(30) (Miss) E. NOETHER (Erlangen): "Ueber rationale Funktionenkörper."

(31) F. NOETHER (Karlsruhe): "Zur Theorie der Turbulenz."

(32) R. KÖNIG (Leipzig): "Arithmetisch-funktionentheoretische Parallelen."

Abstracts of all the papers except the first follow below; the numbers correspond with those in the list of titles above.

2. Professor Müller generalizes the concept of products of points, rays, and planes, as developed by Grassmann, to apply to a number of new combinations which he calls manifold products. Every combination of these products which represents a number is an invariant of the quantities appearing in the equations. They do not depend upon a definite

normal form, a fact that presents many advantages. Finally, the relation between this method and other symbolic methods is pointed out.

3. Nineteen years ago Professor Kohn introduced the throw of v. Staudt as a fundamental concept underlying the projective properties of a geometric configuration. The most important advances made by the same author center about a principle of duality of this geometry, by means of which the throws of figures appear in pairs of reciprocal throws. The contrajective figures or figures of reciprocal throws form a complete analogy to the projective figures or figures of equal throws. In a series of examples the author shows that just those properties have been lacking in which figures of equal throws have been replaced by figures of reciprocal throws.

4. The known relation between the zeros and the linear factors of a binary form was extended by Professor Hočevar to apply to the totality of the points of intersection of a variety $F(x_1, \cdots, x_n) = 0$ and a straight line in R_{n-1}. The process is illustrated geometrically, but the procedure is algebraic.

5. Professor Weitzenböck discussed the following topics: application of symbolic methods of the theory of projective invariants to the invariants of elementary geometry (motion and reflection); proof of the first and second fundamental theorems of the symbolic method, and of the finiteness of a complete system of invariants of motion; construction of such a system for the conic. The memoir is to appear in the *Wiener Berichte.*

6. The latest results of the statistics of insurance, that the probability of death does not agree with that expected from the theory based upon probabilities, makes a new establishment of the foundations of this science necessary. Professor Blaschke proposed a new derivation of the formulas and a new interpretation of the results. A fixed premium is the only equitable method of making payments.

7. In Dr. Du Pasquier's paper we start with a closed company A^1 of insured persons, consisting of persons of the same age x, of the same sex, living contemporaneously in the same

district, in the same occupation, and all married. Let the number of members be $y_1(x)$. In the course of time the number will grow smaller through three causes: death, death of the wife (or husband), invalidity. Those in the second category compose a new company A^{II}, with conditions similar to A^{I}, except that they are not married. Let this number be $y_2(x)$. This number may change by death, marriage, or invalidity. Those in the second set now belong to A^{I} again. A third company A^{III} of $y_3(x)$ persons consists of invalids (unmarried), and a fourth A^{IV} of $y_4(x)$ married invalids. Relations exist between A^{I} and A^{IV} A^{II}, and A^{IV}. These are expressed by means of differential equations of the first order, the constants determined by statistics.

8. In order to establish the theory of gravitation from the standpoint of the principle of relativity, Professor Einstein discussed two possible methods. The first is the principle of the constancy of the velocity of light; it appears as a covariant of linear orthogonal transformations. The simplest theory of this kind is that developed by Nordström: "Gravitations-Feld durch einen Skalar ϕ bestimmt."

In the second method, the velocity of light is assumed to be variable; the gravitation field is determined by a tensor. Instead of four-dimensional vector theory we now have the absolute differential calculus. The equations of the gravitation field are derived from the conservation theorems. They are covariant under any linear transformation.

9. The newly discovered Kepler manuscripts described by Professor v. Dyck concern the publication of the Tabulæ Rudolphinæ and of the Observationes Tychonii Brahe and essays on the problem of the reformation of the calendar. The manuscripts and discussion will soon be published in the *Abhandlungen* of the Bavarian academy of sciences. Professor v. Dyck also explained the new regulation concerning the examination of candidates for higher teachers of mathematics and physics introduced in the Bavarian school system.

10. In his report Professor Korn mentioned the efforts of the committee, composed of prominent mathematicians, astronomers, and physicists, to reduce the nomenclature and definitions of the theory of the potential, to a uniform basis.

In answer to the letter of inquiry, sent to a large number of persons concerned: "What are the words and concepts for which a uniform definition is desirable?" a large number of replies have been received. From these data the committee has discussed the question of the greatest desiderata, and it is believed that by including the theory of elasticity and extending a carefully planned organization gradually uniformity can be accomplished. The committee is confident that in this way an immense service can be accomplished, not only toward understanding the literature, but also toward clearing up a number of grave misunderstandings which now seriously interrupt the further progress of the science.

11. Professor Waelsch's paper is a continuation of his article "Quaternions and binary forms in the Minkowski-Gauss equations of thermodynamics" which has recently appeared in the *Berichte* of the Vienna academy; the present paper will soon appear in the same journal.

12. In the graphical solution of an infinite system of equations with an infinite number of unknowns, Professor Mehmke employed the methods of descriptive geometry, generalized for space of an infinite number of dimensions. The memoir will be published in the *Zeitschrift für Mathematik und Physik*.

13. Professor Juel defines an elementary surface by the property of being divisible into a finite number of parts of surfaces of the third order. It is easily proved that every continuous closed surface having a continuously turning tangent plane will be an elementary surface in case the locus of the point of contact of four-point contact lines divides the surface in a finite number of pieces. For example, every regular analytic surface is an elementary surface. In such cases the ordinary theorems concerning the singular points of plane sections and of contour curves are easily proved geometrically, without making any assumptions concerning derivatives of the second order. It is believed that this procedure will prove to be as fruitful in the study of surfaces as that of the oval and convex arcs has been in the plane.

14. The theory of contact transformations can, as was

shown in the *Wiener Berichte*, volume 110, by S. Kantor, be more easily established by means of the bilinear invariant of the Pfaffian expression $\Sigma p_i du_i$ and the Poisson symbol (φx), the latter to be interpreted as a covariant in plane coordinates. It was shown by Professor Engel that the entire theory of invariants of contact transformations can be expressed by means of any Pfaffian $\alpha_1 du_1 + \cdots + \alpha_{2n} du_{2n}$ that can be put into the normal form $p_1 dx_1 + \cdots + p_{2n} dx_{2n}$.

Moreover, the theory of groups of functions can be given a new interpretation by defining each group by means of the complete system the solution of which consists of the functions, and can all be included, as was also shown by Kantor, among the invariant properties of a linear partial differential equation of the first order in u_{1p} under contact transformation. This problem can also be reduced to Pfaffians.

15. Professor Liebmann reported on the development of contact transformations as follows.

The two sources for the study of contact transformations, systematically elaborated and brought into connection with the theory of continuous groups by Sophus Lie, are the canonical substitutions of mechanics and Plücker's line geometry. The most important lines of investigation at present are in the direction of more minute interpretation and completion on the basis of the geometry of "Speere" (Study), and new methods in the theory of invariants under the infinite group of all point and contact transformations. [See the Mathematical Encyclopedia, Article III *D*7 (German edition now in press).]

16. Consider a one-partite closed, finite space curve which has a single tangent and a single osculating plane at each of its points. Take in particular such a curve of index 4, that is, such that no plane can intersect it in more than four real points. Professor Zindler studies the curve by means of the wedge of a point P on the curve, defined by passing half-planes through the tangent at P and the other points of the curve. The applicability of the idea was shown in the case of curves without cusps, and a set of curves found such that the wedge for each point of the curve is acute. From this fact follows that the curve has four points of undulation and two principal secants, such that the osculating plane at

either point of intersection of the curve and a principal secant must pass through the other. The paper will be published in the *Monatshefte.*

17. Professor Tietze proved that every continuous biuniform mapping of a simply connected surface on itself which preserves the indicatrix is a deformation, that is, that by means of just such representations the identity can be reached.

18. In his paper on continuous mapping, Professor Hahn proved the following theorem: In order that a plane point set M be a continuous representation of a segment, it is necessary and sufficient that it be closed, limited, and connected, and that in each of its points P the condition be fulfilled: to every positive ϵ belongs a positive η such that if a point A of M lies within the region η of P, then in M exists a connected part, containing both P and A and lying within the region ϵ of P.

19. As Dr. Hecke was called away from Vienna during the meeting, his paper was not read.

20. Professor Schrutka's paper was an abstract of a more extensive memoir based on the dissertation of Professor Vahlen, which appeared in *Crelle's Journal*, volume 112, page 1. The purpose of the paper is to introduce simplifications and render more systematic the methods there developed. The entire memoir will soon be published as a monograph.

21. Professor Koebe explained the principle of the method of continuity by means of two examples of conformal depiction of simply and multiply connected regions, and pointed out the differences between the older method of limiting polygons and his own new method of open polygonal continua, founded on theorems of deformation. An extensive application of the principle is made in a memoir being published in the *Mathematische Annalen:* "Ueber die Uniformisierung der algebraischen Kurven IV und V."

22. Professor Plemelj showed that the coefficient a in $f(z) = z + az^2 + \cdots$, in case $f(z)$ is simply represented in the

unit circle, satisfies the inequality $|a| \leqq N < 5$, wherein N is probably equal to 2. The inequality

$$\frac{1}{2N}\left[1-\left(\frac{1-|z|}{1+|z|}\right)^N\right] \leqq |f(z)| < \frac{1}{2N}\left[\left(\frac{1+|z|}{1-|z|}\right)^N - 1\right]$$

can then be derived. In case $N = 2$ we have

$$\frac{|z|}{(1+|z|)^2} \leqq |f(z)| \leqq \frac{|z|}{(1-|z|)^2}$$

as a simple expression of the theorem of deformability of Koebe.

23. In Professor Plemelj's second paper the solutions of the differential equation are expressed as integral functions of accessory parameters, of Laguerre genus zero when the parameters appear linearly. In this way the solutions of very diverse types of differential equations can be obtained, including boundary problems, theorems of oscillation, Klein theorems, etc.

24. Professor Dingeldey first determined the conditions under which the integral

$$\int \frac{x^m}{\sqrt{ax^2+bx+c}}\,dx$$

is algebraic, when m is a positive integer and a, b, c are real. If a, b, c are all different from zero and m even ($m = 2n$), the equation of condition is of order n in b^2 and ac; if $m = 2n+1$, b is a factor, and the other factor is similar to the preceding. By putting $b^2 \div ac = z^2$, the resulting equation in z^2 has only real roots, distinct and positive. It is closely related with the equations appearing in spherical harmonics, which leads to an easy and natural representation by means of a combinant. For example, we have

$$P_5(z) = \frac{1}{5!}\begin{vmatrix} z & 1 & 0 & 0 & 0 \\ 1 & 3z & 2 & 0 & 0 \\ 0 & 2 & 5z & 3 & 0 \\ 0 & 0 & 3 & 7z & 4 \\ 0 & 0 & 0 & 4 & 9z \end{vmatrix},$$

and similar expressions for $P_m(z)$.

An extensive abstract will be published in the *Jahresbericht.*

25. What conditions must a curvilinear integral

$$\int \left(Pdx + \sum_{k=1}^{n} Q_k dy_k \right)$$

satisfy in order to be independent of the path of integration when the paths are restricted by certain differential equations

$$\varphi_a(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0, \quad (a = 1, \cdots, m < n)?$$

In order to answer this question Dr. Radon assumes that the differential equations do not possess an integral $\phi(x, y_1, \cdots, y_n) = c$. The important results are: If by means of the $\varphi_a = 0$ no linear equation $adx + \Sigma b_k dy_k = 0$ can be derived, then $Pdx + \Sigma Q_k dy_k$ must be a complete differential. If, on the other hand, such linear equations can be derived, then the expression $Pdx + \Sigma Q_k dy_k$, after simplification by means of these equations, is reduced to a complete differential in the ordinary sense.

26. In a previous memoir, published in the *Mathematische Annalen,* Dr. Gross considered a system of differential equations which proved to be non-integrable, that is, unsolvable by means of the processes of differentiation and elimination, and the addition of arbitrary functions. In this memoir he shows how one can determine whether a given system belongs to the class of those previously discussed, and also how a solution (if there is any) can be obtained.

27. In establishing the method of least squares, in so far as the proof is not a variation of that given by Gauss, the attempt is made to select the most favorable linear combination from among the possible combinations of error equations. It is by no means evident why restriction to linear equations is made. Professor Suppantschitsch shows how this restriction is connected with simpler assumptions (axioms). The only quality of the functions employed that is required is continuity. The paper is being published in the *Wiener Berichte.*

28. In his correspondence with Jacques de Billy, Fermat

takes up the question, handed down from Diophantos, to find an infinite number of pairs of rational numbers (x, y) which satisfy the equation $y^3 = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$, the coefficients being rational. Professor Haentzschel discussed the derivation of the conditions under which the problem can be solved. The equation is satisfied if the equation $v^2 = 4s^3 + D$ has an infinite number of solutions, D being the discriminant of the cubic form. This is the first condition. The second condition is that the equation $v = -c_{3,3}(x)/f(x)$ shall have a rational root (Haentzschel: "Ueber das Integral . . .," *Berlin Mathematische Gesellschaft*, 1910). This root is the first solution. The further investigation consists of three parts: (*a*) D any rational number; (*b*) D a rational square; (*c*) $D = 0$. In each case there are two subdivisions, according as a_0 is or is not a cube.

29. Professor Perron showed that in order to obtain the complete integral of a differential equation of the form

$$\left[\frac{dy}{dx} - f(x, y)\right]^2 - [\varphi(x, y)]^2 = 0$$

it is not sufficient to integrate the two equations

$$\frac{dy}{dx} - f + \varphi = 0, \qquad \frac{dy}{dx} - f - \varphi = 0,$$

as has always been supposed. For example, the equation

$$\left(\frac{dy}{dx}\right)^2 - \left(x \sin\frac{1}{x}\right)^2 = 0$$

has, besides the two integrals

$$y_1 = \int_0^x x \sin\frac{1}{x}\,dx, \qquad y_2 = -\int_0^x x \sin\frac{1}{x}\,dx,$$

both of which vanish with x, an infinite number of others, including

$$y_3 = \int_0^x \left| x \sin\frac{1}{x} \right| dx.$$

30. The treatment of fields of rational functions, as de-

veloped by Dr. Emmy Noether, concerns fields whose elements are rational functions of n variables. For these fields, the existence of a rational basis is proved, and the minimum basis defined. An application of the latter is made to the problem of constructing equations with a preassigned group. In conclusion, the existence of a basis of integrity is proved for a class of fields.

31. The hydrodynamic treatment of the phenomena of turbulence requires the determination of further stationary or of periodic solutions of the hydrodynamic boundary problem in addition to the known laminal flow. The treatment heretofore employed is confined to the immediate region of this flow and does not lead to the other real solutions. Dr. F. Noether showed that it is necessary to consider solutions having a finite variation from the laminal flow. The theory of the ramification of non-linear integral equations leads to the hypothesis that with a sufficiently large value of the Reynolds number such solutions ought to exist.

By means of Fourier series, the problem can be reduced to a boundary problem of ordinary non-linear differential equations. It is thus brought into close relation with a previous memoir of the author, discussed in the *Berichte der Bayerischen Akademie*, 1913, page 309.

32. Dr. König pointed out the analogies which exist between elementary arithmetic (theory of real numbers), advanced arithmetic (algebraic numbers) and the Riemannian theory of functions, and showed that the concepts of irrational numbers, ideals, and multiplicative functions are all extensions of a single idea. A more detailed abstract will appear in the next number of the *Jahresbericht*.

VIRGIL SNYDER.

ON BINARY MODULAR GROUPS AND THEIR INVARIANTS.

BY PROFESSOR L. E. DICKSON.

A VERY simple determination is here made of all groups G of binary linear transformations with integral coefficients of determinant unity modulo p such that the order of the group is divisible* by the prime p. The corresponding problem for which the coefficients are in any finite field of order a power of p has been treated recently by H. H. Mitchell,† who cites the earlier treatments by Gierster, E. H. Moore, Wiman, and Dickson of the related linear fractional group. To be added to these references is a paper on the present binary homogeneous groups.‡

Any binary modular transformation T multiplies some linear function of x and y by a constant. This constant is unity if T is of period p. Hence after a suitable choice of the variables, we may assume that our group G contains

$$T: \quad x' \equiv x + y, \quad y' \equiv y \pmod{p}.$$

THEOREM. *Either G is the group Γ of all binary transformations with integral coefficients of determinant unity modulo p, or else every transformation of G is of the form*

$$(1) \qquad x' \equiv tx + ly, \quad y' \equiv t^{-1}y \pmod{p}.$$

Suppose that G contains a transformation R for which y' involves x, so that $y' = a(x + ky)$, $a \not\equiv 0$. Then G contains§

$$P = R^{-1}T^k: \quad x' \equiv y/a, \quad y' \equiv -ax + by \pmod{p},$$

in which the value of b is immaterial. Next, G contains

$$T^{b/a}P: \quad x'' \equiv y/a, \quad y'' \equiv -ax.$$

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* The groups of orders prime to p may be found as in the case of binary collineation groups.

† *Trans. Amer. Math. Soc.*, vol. 12 (1911), p. 207.

‡ Dickson, "Binary modular groups and their invariants," *Amer. Jour. Math.*, vol. 33 (1911), p. 175. Here are found the invariants of any modular group other than one composed only of transformations (1), the case treated in the text.

§ To form the product RP, we note that $x' = a^{-1}y'$ under P and eliminate y' by means of the equation for R. But $x'' \equiv x + ky$ under T^k.

The latter transforms T into

$$x' \equiv x, \quad y' \equiv y - a^2x.$$

A power of this is $x' \equiv x$, $y' \equiv y + x$. The latter and T are known to generate the group Γ. Hence the theorem is proved.

The group G of transformations (1) is generated by T and

$$x' \equiv \tau x, \quad y' \equiv \tau^{-1}y, \tag{2}$$

where τ belongs to a certain exponent d modulo p. Thus G is of order pd.

Evidently T leaves absolutely unaltered the product

$$\lambda = x(x+y)(x+2y) + \cdots + (x+\overline{p-1}y) \equiv x^p - xy^{p-1} \pmod{p}, \tag{3}$$

the congruence holding in view of Fermat's theorem. Now (2) replaces λ by $\tau\lambda$. Thus G has the relative invariants λ and y.

If (1) leaves the point (x, y) unaltered,

$$tx + ly \equiv \rho x, \quad t^{-1}y \equiv \rho y \pmod{p}. \tag{4}$$

If these congruences hold identically, $l \equiv 0$, $t \equiv \pm 1$, and (1) becomes

$$x' \equiv \pm x, \quad y' \equiv \pm y. \tag{5}$$

First, let d be even and $p > 2$. Then G contains the two transformations (5), which leave every point unaltered. A point is called special if it can not be transformed by G into $pd/2$ distinct points, and hence is unaltered by some transformation (1) not of type (5). For such a transformation, equations (4) are not both identities and determine uniquely x/y as an integer modulo p, and hence a real special point (x, y). But $(1, 0)$ is unaltered by G, while the remaining real points $(k, 1)$ are permuted by the powers of T. Thus any invariant which vanishes at a special point has the factor y or λ. An invariant without a factor y or λ therefore vanishes at imaginary points falling into sets of $pd/2$ points conjugate under G. Now $y^{pd/2}$ and $\lambda^{d/2}$ are unaltered by T and changed in sign by (2), since $\tau^{\pm d/2} \equiv -1 \pmod{p}$. Hence any linear combi-

nation of them is an invariant of G. We can find* a product of such combinations which has integral coefficients and vanishes at any assigned point, not a special point. Thus the invariant is the product of one or more such products.

For d odd, a non-special point is one of pd conjugates under G. We now use the absolute invariants y^{pd}, λ^d.

THEOREM. *As a fundamental system of invariants of a group of transformations* (1), *we may take* y *and* λ.

In particular, this theorem yields the seminvariant leaders of invariants of two pairs of cogredient variables.

UNIVERSITY OF CHICAGO,
February, 1913.

ON SOME SYSTEMS OF COLLINEATION GROUPS.

BY DR. HOWARD H. MITCHELL.

(Read before the American Mathematical Society, April 26, 1913.)

§ 1.

SOME systems of collineation groups which arise in connection with the theory of elliptic functions have been investigated by Klein† and Hurwitz‡. One of them is a system in n variables each group of which contains an invariant subgroup of order n^2. For n a prime the quotient group with respect to this invariant subgroup is (1, 1) isomorphic with the modular group on two indices of order $n(n^2-1)$. The group in three variables is the Hessian group of order 216.

For n odd there is also an invariant subgroup of order $2n^2$, and there exist two other groups in $(n-1)/2$ and $(n+1)/2$ variables each of which is isomorphic with the quotient group with respect to this subgroup. Thus for $n=5$ there is both a binary and a ternary G_{60} and for $n=7$ both a ternary and a quaternary G_{168}.

Similar systems of groups in n^2, $(n^2-1)/2$, and $(n^2+1)/2$ variables which arise in the theory of hyperelliptic functions

* Dickson, *Trans. Amer. Math. Soc.*, vol. 12 (1911), p. 4.
† *Math. Annalen*, vol. 15 (1879), p. 275; also Klein-Fricke, Modulfunktionen (2) 5.
‡ *Math. Annalen*, vol. 27 (1885), p. 198.

have been investigated by Witting* and Burkhardt.† The group in n^2 variables for n a prime is isomorphic with the abelian linear group on four indices with modulus n.

For n odd there is an invariant subgroup of order 2, and the groups in $(n^2-1)/2$ and $(n^2+1)/2$ variables are (1, 1) isomorphic with the quotient group with respect to this subgroup. Thus for $n=3$ there is both a quaternary and a quinary group of order 25920.

The groups in n and n^2 variables, when n is a prime p, are closely connected with a group in p^m variables investigated by Jordan‡ and Dickson.§ As a collineation group it contains an invariant subgroup of order p^{2m} and the quotient group is (1, 1) isomorphic with the abelian linear group on $2m$ indices and coefficients reduced modulo p. For p odd there is an invariant subgroup of order $2p^{2m}$ and there exist two other groups in $(p^m-1)/2$ and $(p^m+1)/2$ variables which are isomorphic with the quotient group with respect to this subgroup. These two groups do not seem to have been noticed before for $m>2$.

All three of these groups are shown by the author to be "transitive" and "primitive" in the variables. These terms in relation to collineation groups seem to have been employed first by Maschke and Blichfeldt.

§ 2.

Jordan and Dickson consider p^m variables which are distinguished by m subscripts, $\xi_1, \xi_2, \cdots, \xi_m$, each of which is an integer reduced modulo p. This may readily be translated into ordinary numbering if we consider that $\xi_1, \xi_2, \cdots, \xi_m$ represents $\xi_1+\xi_2 p+\cdots+\xi_m p^{m-1}$, where the smallest positive (or 0) residue is taken.

The group in these variables which they consider is generated by the following operators:

$$A_i: \quad X_{\xi_1\cdots\xi_m}' = \theta^{\xi_i} X_{\xi_1\cdots\xi_m};$$

$$B_i: \quad X_{\xi_1\cdots\xi_i\cdots\xi_m}' = X_{\xi_1\cdots\xi_i+1\cdots\xi_m};$$

$$L_i: \quad X_{\xi_1\cdots\xi_m}' = \theta^{\frac{1}{2}\xi_i(\xi_i+p)} X_{\xi_1\cdots\xi_m};$$

* *Math. Annalen*, vol. 29 (1886), p. 157; Diss. (Göttingen, 1887).
† *Math. Annalen*, vol. 38 (1891), p. 163.
‡ Traité des Substitutions, pp. 420–450.
§ *Trans. Amer. Math. Soc.*, vol. 1, pp. 30–38.

$$M_i: \quad X_{\xi_1 \cdots \xi_i \cdots \xi_m}{}' = \frac{(-1)^{r/2}}{\sqrt{p}} \sum_{\mu=0}^{p-1} \theta^{\mu \xi_i} X_{\xi_1 \cdots \mu \cdots \xi_m};$$

$$N_{ij}: \quad X_{\xi_1 \cdots \xi_m}{}' = \theta^{\xi_i \xi_j} X_{\xi_1 \cdots \xi_m};$$

where $i, j = 1, 2, \cdots, m$; $j \neq i$; $\theta^p = 1$, $\theta \neq 1$; and r is so chosen that the determinant of M_i is unity. If $p = 2$ the exponent of θ in L_i is not reduced modulo 2. The multiplier is then $-\sqrt{-1}$ if $\xi_i \equiv 1$, and $+1$ if $\xi_i \equiv 0$, mod 2. In this case L_i is of period 4, and for p odd it is of period p.

The group leaves invariant the subgroup of order* p^{2m} generated by the $2m$ operators, A_i, B_i. If any operator of this subgroup be expressed in the form

$$\prod_{i=1}^{m} A_i^{x_i} B_i^{y_i},$$

it is shown by Jordan that the substitutions on the $2m$ indices x_i, y_i which are effected when L_i, M_i, N_{ij} are used to transform this general operator, generate the abelian linear group† on those $2m$ indices. The operators L_i as written here are equal to the L_i of Jordan by a power of A_i, but since A_i leaves all the indices unaltered, the resulting substitution is the same.

If $p > 2$, the group on these $2m$ indices contains an invariant substitution which changes the sign of each of them. One of the collineations which effects this substitution on the $2m$ indices is

$$R: \quad X_{\xi_1 \cdots \xi_m}{}' = (-1)^{\frac{1}{2}(p^m-1)} X_{-\xi_1 \cdots -\xi_m}.$$

This transformation is commutative with L_i, M_i, and N_{ij} and is transformed into p^{2m} conjugate operators by the invariant subgroup. Hence the group commutative with R is (1, 1) isomorphic with the abelian linear group.

It will be observed that R multiplies by ± 1 each of the $(p^m + 1)/2$ linear expressions

$$X_{0 \cdots 0}, \quad X_{\xi_1 \cdots \xi_m} + X_{-\xi_1 \cdots -\xi_m},$$

and multiplies by ∓ 1 each of the $(p^m - 1)/2$ expressions

$$X_{\xi_1 \cdots \xi_m} - X_{-\xi_1 \cdots -\xi_m},$$

* As a *linear* group this subgroup is of order p^{2m+1} if p is odd and 2^{2m+2} if $p = 2$; in the latter case an additional generator being necessary.

† L. c.; also Dickson, Linear Groups, p. 92.

where not every ξ_i is 0. Each of the transformations L_i, M_i, N_{ij} therefore transforms any one of these expressions into a linear combination of the others in the same set. Hence there exist collineation groups in both $(p^m + 1)/2$ and $(p^m - 1)/2$ variables which are isomorphic with the abelian linear group. Since the quotient group of that group with respect to its invariant operator of period 2 is simple except in the case $p^m = 3^1$, it follows that the group in each set of variables as a collineation group is (1, 1) isomorphic with this quotient group.

§3.

It may be shown that each of these three groups is "transitive" and "primitive" in the variables on which it is represented. By a transitive collineation group is meant one in which the matrix of the coefficients of every conjugate set of linear expressions is not 0. It follows from the work of Maschke* that an intransitive group is completely reducible, i. e., that the variables may be separated into sets such that each variable is replaced by a linear combination of those in the same set by all the transformations of the group. If the group is also primitive in the sense in which Blichfeldt uses the term, there exist no systems of intransitivity which are permuted by the group.

The group in the p^m variables is readily seen to be transitive. The only linear expressions left invariant except for a multiplier by each of the transformations A_i are the p^m variables themselves. Since the group generated by the A's is commutative as a linear group as well as a collineation group, any linear system of the variables containing k homogeneous parameters which is left unaltered must contain k expressions which are left individually unaltered except for a multiplier, i. e., k of the variables themselves. But no such set is left invariant by all the B's. Hence the invariant subgroup is itself transitive in the variables.

We consider now the group commutative with R. If two variables with subscripts $\xi_1\xi_2 \cdots \xi_m$ and $\eta_1\eta_2 \cdots \eta_m$ have the same multiplier under each of the transformations L_i, N_{ij}, we must have $\xi_i^2 = \eta_i^2$, $\xi_i\xi_j = \eta_i\eta_j$ $(i, j = 1, 2, \cdots, m)$. Hence $\xi_i = \pm \eta_i$, where the sign is the same for all values of i.

* *Math. Annalen*, vol. 52 (1899), p. 363.

Hence of the expressions

$$X_{0\cdots 0}, \quad X_{\xi_1\cdots\xi_m} + X_{-\xi_1\cdots-\xi_m}, \quad X_{\xi_1\cdots\xi_m^1} - X_{-\xi_1\cdots-\xi_m},$$

no two in either set have the same multiplier under all the transformations L_i, N_{ij}.

Hence if there is any system of intransitivity in either set under the whole group commutative with R, it must be determined by a certain number of these expressions themselves. But under the product $M_1M_2\cdots M_m$, $X_{0\cdots 0}$ is transformed into the sum of all the p^m variables. Hence the group in the $(p^m+1)/2$ variables is transitive.

Under $M_1M_2\cdots M_m$, $X_{10\cdots 0} - X_{-10\cdots 0}$ is transformed into a linear expression involving all the variables in which $\xi_1 \neq 0$. Under $M_2\cdots M_m$, $X_{010\cdots 0} - X_{0-10\cdots 0}$ is transformed into a linear expression involving all the variables in which $\xi_1 = 0$, $\xi_2 \neq 0$. In general, under $M_i\cdots M_m$, $X_{0\cdots\xi_i\cdots 0} - X_{0\cdots-\xi_i\cdots 0}$ is transformed into an expression involving all the variables in which $\xi_1 = \cdots = \xi_{i-1} = 0$, $\xi_i \neq 0$.

But these m sets of variables are transformed among themselves. For under M_1, $X_{0\cdots\xi_i\cdots 0} - X_{0\cdots-\xi_i\cdots 0}$ is transformed into an expression involving some variables in which $\xi_1 \neq 0$. Hence the group in $(p^m-1)/2$ variables is transitive.

All three systems of groups are readily seen to be primitive in the variables on which they are represented. For if there exist systems of intransitivity, each L_i must permute some of them in cycles of period p if p is odd and 2 or 4 if $p=2$. If p is odd this is impossible, since each L_i has only $(p+1)/2$ different multipliers and hence cannot transform a linear expression in the variables into p linearly independent expressions. For the same reason it cannot be of period 4 on any systems of intransitivity if $p=2$. But if it is of period 2 on them, its square will leave them all invariant. Since this square belongs to the invariant subgroup of order 2^{2m}, that subgroup would have to be intransitive, since all its operators of period 2 are conjugate. But we have already seen that that subgroup is transitive.

UNIVERSITY OF PENNSYLVANIA.

ON THE SUMMABILITY OF FOURIER'S SERIES.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society, February 22, 1913.)

1. LET

$$A_n^{(k)} = \frac{(k+1)(k+2)\cdots(k+n)}{n!}$$

(1)

$$= \frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+1)} \quad (n=1,2,3,\cdots), \quad A_0^{(k)}=1,$$

so that

$$\frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n, \quad (|z|<1); \tag{2}$$

then the identity

$$\sum_{n=0}^{\infty} A_n^{(k)} z^n = \frac{1}{(1-z)^{k+1}} = \frac{1}{1-z}\cdot\frac{1}{(1-z)^k} = \sum_{\mu=0}^{\infty} z^\mu \cdot \sum_{\nu=0}^{\infty} A_\nu^{(k-1)} z^\nu$$

gives

$$A_n^{(k)} = \sum_{\nu=0}^{n} A_\nu^{(k-1)} = \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}. \tag{3}$$

The nth Cesàro mean of order k of a given series $u_0 + u_1 + \cdots + u_n + \cdots$ is, by definition, equal to

$$s_n^{(k)} = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k)} u_\nu = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} u_\mu \tag{4}$$

(both definitions being equivalent on account of (3)), and if $\lim_{n=\infty} s_n^{(k)}$ exists and equals s, the given series is said to be summable by Cesàro's means of order k, or briefly, summable (Ck), with the sum s.

In the present note, I propose to give a simplified proof of the following theorem, due to Riesz and Chapman:*

* M. Riesz, "Sur les séries de Dirichlet et les séries entières," *Comptes rendus de l'Académie des Sciences* (Paris), vol. 149 (1909), pp. 909–912 (gives no details of the proof).

S. Chapman, "Non-integral orders of summability of series and integrals," *Proceedings of the London Mathematical Society*, ser. 2, vol. 9 (1911), pp. 369–409. (See p. 390.)

Let $f(x)$ be a function defined in the interval $-\pi \leqq x \leqq \pi$, and such that in this interval $|f(x)|$ is integrable in the Lebesgue sense; then the Fourier series for $f(x)$ is summable $(C k)$ for any $k > 0$ with the sum $\frac{1}{2}(f(x+0) + f(x-0)) = \frac{1}{2}\lim_{\epsilon=0}(f(x+\epsilon)+f(x-\epsilon))$ at any point where this limit exists.† The convergence of the Cesàro means of order k towards this limit is uniform on any closed range for every point of which $f(x)$ is bounded and $f(x+0)+f(x-0)$ exists uniformly.*

2. To prove this theorem, we start from the well known expression for the sum of the $n+1$ first terms of the Fourier series for $f(x)$.

$$s_n\{f(x)\} = \frac{1}{\pi}\int_{-(\pi+x)/2}^{(\pi-x)/2} f(x+2y)\frac{\sin(2n+1)y}{\sin y}dy$$

$$= \frac{1}{\pi}\int_{-(\pi/2)}^{\pi/2} f(x+2y)\frac{\sin(2n+1)y}{\sin y}dy$$

on account of the periodicity of $f(x)$, as established in footnote *, and this expression is easily transformed into

$$s_n\{f(x)\} = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y)+f(x-2y))\frac{\sin(2n+1)y}{\sin y}dy,$$

and the nth Cesàro mean of order k of the Fourier series in question becomes, by (4),

$$s_n^{(k)}\{f(x)\} = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y)+f(x-2y))s_n^{(k)}(y)dy,$$

The simplification in the present proof lies in the method of arriving at the inequality (7), which is obtained by Chapman by a method equivalent to the application of Euler's summation formula to the expression.

The present method is applicable also to the corresponding problem in the expansion of a function of two variables in a series of spherical harmonics; see my forthcoming papers in the *Mathematische Annalen:* "Über die Laplace'sche Reihe" and "Über die Summirbarkeit der Reihen von Laplace und Legendre."

* In Chapman's statement of the theorem, $f(x)$ is only required to be integrable in the Lebesgue sense without being absolutely integrable (both requirements being equivalent only when $f(x)$ is bounded for $-\pi \leqq x \leqq \pi$). In Art. 3 of the present note, it is shown by an example that in this form the theorem is not generally true.

† For $x = \pm\pi$, this limit should be replaced by $\frac{1}{2}f[(-\pi+0)+f(\pi-0)]$, which may be included in the expression above by defining $f(x)$ outside of the interval $-\pi \leqq x \leqq \pi$ as periodic with the period 2π.

where

$$(5) \qquad s_n^{(k)}(y) = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \frac{\sin(2\nu+1)y}{\sin y}.$$

Making $f(x) = 1$, we obtain

$$1 = s_n^{(k)}\{1\} = \frac{2}{\pi}\int_0^{\pi/2} s_n^{(k)}(y)dy,$$

and consequently

$$(6) \qquad \begin{aligned} &s_n^{(k)}\{f(x)\} - \tfrac{1}{2}(f(x+0)+f(x-0)) \\ &= \frac{1}{\pi}\int_0^{\pi/2} (f(x+2y)+f(x-2y)-f(x+0) \\ &\qquad -f(x-0))s_n^{(k)}(y)dy = \frac{1}{\pi}\int_0^{\epsilon} + \frac{1}{\pi}\int_{\epsilon}^{\pi/2} \end{aligned}$$

where $0 < \epsilon < \pi/4$.

We now assume $k < 1$;* the main point in our proof consists in showing that

$$(7) \qquad \frac{1}{\pi}\int_0^{\pi/4} |s_n^{(k)}(y)|\,dy < c_1 \quad (n = 2, 3, 4, \cdots),$$

where c_1, as well as c_2, c_3, $\cdots$ which will be introduced later, are positive constants independent of n. We decompose our integral as follows:

$$\frac{1}{\pi}\int_0^{\pi/4} |s_n^{(k)}(y)|\,dy = \frac{1}{\pi}\int_0^{\pi/(2n+1)} + \frac{1}{\pi}\int_{\pi/(2n+1)}^{\pi/4}.$$

As we have

$$\left|\frac{\sin(2\nu+1)y}{\sin y}\right| = \left|1 + 2\sum_{\mu=1}^{\nu} \cos 2\mu y\right| \leqq 2\nu+1 \leqq 2n+1,$$

it follows that

$$|s_n^{(k)}(y)| \leqq \frac{1}{A_n^{(k)}} \cdot (2n+1)\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} = 2n+1,$$

and consequently

$$(8) \qquad \frac{1}{\pi}\int_0^{\pi/(2n+1)} |s_n^{(k)}(y)|\,dy < \frac{1}{\pi}\int_0^{\pi/(2n+1)} (2n+1)dy = 1.$$

* A series being (uniformly) summable (Ck) is also (uniformly) summable (Ck') with the same sum when $k' > k$ (see Chapman, l. c.), and it is therefore sufficient to prove our theorem for $k < 1$.

To estimate the second part of our integral, we observe that, for $|z| < 1$,

$$\frac{1}{1 - ze^{2\nu i}} = \sum_{n=0}^{\infty} z^n e^{2n\nu i},$$

$$\frac{1}{(1-z)^k(1 - ze^{2\nu i})} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n \cdot \sum_{n=0}^{\infty} z^n e^{2n\nu i}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu\nu i},$$

or writing $1/z$ instead of z,

$$\frac{z^{k+1}}{(z-1)^k(z - e^{2\nu i})} = \sum_{n=0}^{\infty} z^{-n} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu\nu i},$$

whence, by Cauchy's theorem,

$$(9) \qquad \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu\nu i} = \frac{1}{2\pi i} \int_C \frac{z^{n+k} dz}{(z-1)^k(z - e^{2\nu i})},$$

the integration being performed in the positive sense over a contour C enclosing the points $z = 1$ and $z = e^{2\nu i}$, and the determinations of z^k and $(z-1)^k$ being taken so that they are real and positive for z real and > 1. We now deform the contour C into a circuit C_1 consisting of (1) the straight line from $z = 0$ to $z = 1 - \eta$, where $\eta > 0$; (2) the circle $z = 1 + \eta e^{\theta i}$, $-\pi \leqq \theta \leqq \pi$; and (3) the straight line from $z = 1 - \eta$ to $z = 0$, followed by a similar circuit C_2 around $z = e^{2\nu i}$. As $0 < k < 1$, the integral over (2) tends towards zero with η; on (1) and (3) we have $z^k > 0$, and as $(z-1)^k > 0$ for $z = 1 + \eta$, we have $(z-1)^k = e^{-k\pi i}(1-z)^k$ on (1), but $(z-1)^k = e^{k\pi i}(1-z)^k$ on (3), so that, letting η tend towards zero,

$$\frac{1}{2\pi i}\int_{C_1} = \frac{1}{2\pi i}\int_0^1 \frac{z^{n+k} dz}{e^{-k\pi i}(1-z)^k(z - e^{2\nu i})} + \frac{1}{2\pi i}\int_1^0 \frac{z^{n+k} dz}{e^{k\pi i}(1-z)^k(z - e^{2\nu i})}$$

$$= \frac{\sin k\pi}{\pi} \int_0^1 \frac{z^{n+k} dz}{(1-z)^k(z - e^{2\nu i})}.$$

We also have

$$\frac{1}{2\pi i}\int_{C_2} = \frac{e^{2(n+k)yi}}{(e^{2yi}-1)^k} = \frac{e^{(2n+k)yi+(k\pi i/2)}}{(2\sin y)^k},$$

this being the residue of the integrand at $z = e^{2yi}$. Denoting by M the minimum of $|z - e^{2yi}|$ for $0 \leqq z \leqq 1$, so that

$$M = \begin{cases} \sin 2y & \left(0 < 2y \leqq \frac{\pi}{2}\right), \\ 1 & \left(\frac{\pi}{2} \leqq 2y \leqq \pi\right), \end{cases} \tag{10}$$

we then obtain from (9)

$$\left|\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi}\right| < \frac{1}{\pi M}\int_0^1 z^{n+k}(1-z)^{-k}dz + \frac{1}{(2\sin y)^k}$$

$$= \frac{1}{\pi}\frac{\Gamma(1-k)\Gamma(n+k+1)}{\Gamma(n+2)}\cdot\frac{1}{M} + \frac{1}{(2\sin y)^k},$$

and consequently

$$\begin{aligned} |s_n^{(k)}(y)| &= \frac{1}{A_n^{(k)}\sin y}\left|\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}\sin(2\nu+1)y\right| \\ &\leqq \frac{1}{A_n^{(k)}\sin y}\left|e^{yi}\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi}\right| \\ &< \frac{1}{\pi}\frac{\Gamma(1-k)\Gamma(n+k+1)}{\Gamma(n+2)A_n^{(k)}}\cdot\frac{1}{M\sin y} + \frac{2}{A_n^{(k)}}\cdot\frac{1}{(2\sin y)^{k+1}} \\ &= \frac{\Gamma(1-k)\Gamma(1+k)}{\pi}\cdot\frac{1}{n+1}\cdot\frac{1}{M\sin y} + \frac{2}{A_n^{(k)}}\cdot\frac{1}{(2\sin y)^{k+1}}. \end{aligned} \tag{11}$$

By Stirling's formula, it is readily seen from (1) that

$$\frac{2}{A_n^{(k)}} < \frac{c_2}{(n+1)^k}, \tag{12}$$

and from (10), (11) and (12) we obtain, for $0 < y \leqq \pi/4$,

$$|s_n^{(k)}(y)| < \frac{\Gamma(1-k)\Gamma(1+k)}{\pi} \cdot \frac{1}{n+1} \cdot \frac{1}{\sin y \sin 2y} + \frac{c_2}{(n+1)^k} \cdot \frac{1}{(2\sin y)^{k+1}}$$

$$\leqq \frac{\Gamma(1-k)\Gamma(1+k)}{\pi} \cdot \frac{1}{n+1} \cdot \frac{1}{\frac{2}{\pi}y \cdot \frac{2}{\pi} \cdot 2y} + \frac{c_2}{(n+1)^k} \cdot \frac{1}{\left(2 \cdot \frac{2}{\pi}y\right)^{k+1}}$$

$$= \frac{c_3}{n+1} \cdot \frac{1}{y^2} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{y^{k+1}},$$

whence

$$\frac{1}{\pi}\int_{\pi/(2n+1)}^{\pi/4} |s_n^{(k)}(y)|\,dy < \frac{1}{\pi}\int_{\pi/(2n+1)}^{\pi/4} \left(\frac{c_3}{n+1} \cdot \frac{1}{y^2} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{y^{k+1}}\right) dy$$

$$< \frac{1}{\pi}\left(\frac{c_3}{n+1} \cdot \frac{2n+1}{\pi} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{k} \cdot \frac{(2n+1)^k}{\pi^k}\right) < c_5,$$

from which inequality and (8) we immediately deduce (7).

For $0 < \epsilon < \pi/4$, we also obtain from (10), (11) and (12)

$$|s_n^{(k)}(y)| < \frac{c_6}{(n+1)^k} \cdot \frac{1}{\sin^2 \epsilon} \qquad \left(\epsilon \leqq y \leqq \frac{\pi}{2}\right). \tag{13}$$

In (6), make ϵ so small that, δ being a given positive quantity,

$$|f(x+2y) + f(x-2y) - f(x+0) - f(x-0)| < \frac{\delta}{2c_1} \qquad \left(0 \leqq y \leqq \epsilon < \frac{\pi}{4}\right); \tag{14}$$

then

$$\left|\frac{1}{\pi}\int_0^\epsilon (f(x+2y)+f(x-2y)-f(x+0)-f(x-0))s_n^{(k)}(y)\,dy\right| < \frac{1}{\pi}\int_0^\epsilon \frac{\delta}{2c_1}|s_n^{(k)}(y)|\,dy < \frac{\delta}{2c_1} \cdot \frac{1}{\pi}\int_0^{\pi/4} |s_n^{(k)}(y)|\,dy = \frac{\delta}{2}. \tag{15}$$

On account of (13), we also have, bearing in mind the absolute integrability of $f(x)$,

$$\left|\frac{1}{\pi}\int_{\epsilon}^{\pi/2}(f(x+2y)+f(x-2y)-f(x+0)-f(x-0))s_n^{(k)}(y)dy\right|$$
$$<\frac{c_6}{(n+1)^k}\cdot\frac{1}{\sin^2\epsilon}\cdot\frac{1}{\pi}\int_{\epsilon}^{\pi/2}|f(x+2y)+f(x-2y)-f(x+0)$$
$$(16)\quad -f(x-0)|\,dy<\frac{c_6}{(n+1)^k}\cdot\frac{1}{\pi\sin^2\epsilon}\left[\int_{\epsilon}^{\pi/2}|f(x+2y)|\,dy\right.$$
$$\left.+\int_{\epsilon}^{\pi/2}|f(x-2y)|\,dy+|f(x+0)-f(x-0)|\left(\frac{\pi}{2}-\epsilon\right)\right]$$
$$<\frac{c_7}{(n+1)^k\sin^2\epsilon}.$$

After fixing an ϵ satisfying (14), we determine an $N=N(\epsilon)$ so large that (16) becomes less than $\delta/2$ for $n\geqq N$, and (6), (15) and (16) give

$$|\,s_n^{(k)}\{f(x)\}-\tfrac{1}{2}(f(x+0)+f(x-0))\,|<\delta \quad \text{for} \quad n\geqq N,$$

which proves the first part of the theorem. In regard to the second part, it is sufficient to observe that, the range in question being closed, an ϵ and a c_1 may be determined independent of x so that (14) and (16) hold *uniformly* over the range in question.

3. To show that the theorem is not generally true when $f(x)$ is integrable without being absolutely integrable, consider the function of period 2π defined by

$$f(x)=\frac{d}{dx}\left(x^{\nu}\cos\frac{1}{x}\right) \qquad (0\leqq x\leqq 2\pi).$$

Riemann* has shown that, for $0<\nu<\frac{1}{2}$, the nth term in the Fourier series corresponding to this function has the asymptotic expression

$$\left(\frac{1}{2\sqrt{\pi}}\sin\left(2\sqrt{n}-nx+\frac{\pi}{4}\right)+\epsilon_n\right)n^{(1-2\nu)/4},\quad \lim_{n=\infty}\epsilon_n=0,$$

* B. Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," Gesammelte Werke, second edition (Leipzig, 1892), pp. 227–265. See pp. 260 et seq.

and, as for the summability (Ck) of the series $u_0 + u_1 + \cdots + u_n + \cdots$ it is necessary that*

$$\lim_{n=\infty} \frac{u_n}{n^k} = 0,$$

it follows that, for any $k < \frac{1}{4}$, we obtain a Fourier series which is not summable (Ck) for any value of x by selecting a ν such that $1 - 2\nu > 4k$. By a suitable modification of Riemann's example, we may construct a Fourier series with the corresponding property for any $k < \frac{1}{2}$; for $1 > k \geqq \frac{1}{2}$, I have not been able to decide whether the theorem is true for all integrable (and not only absolutely integrable) functions or not.†

CHICAGO, ILL.,
February 3, 1913.

NOTE ON PIERPONT'S THEORY OF FUNCTIONS.

IN a review, written some years ago, of Pierpont's Theory of Functions of Real Variables, I made the following incorrect statement with regard to the possibility of reversing the order of differentiation of a function $f(x, y)$:‡

"Under the assumption that f_x' exists on $y = b$, f_y' on $x = a$, and that one of them is approached uniformly, it follows as a corollary to the theorem of Moore mentioned above, that the second derivatives f_{xy}'', f_{yx}'' exist at (a, b) and are equal."

The assumptions should be that f_x' exists on $x = a$, f_y' on $y = b$, and that the derivative for x at $x = a$ of the quotient $f(x, y)/(y - b)$ is approached uniformly for values of y different from b. These are the hypotheses, in different words, which Professor E. H. Moore uses in the Lectures referred to on page 124 of the review, and which I intended to reproduce.

I am indebted for this correction to Mr. G. A. Pfeiffer. In a recent letter to me he cited the example $f = xy(x^2 - y^2)/(x^2 + y^2)$ with the agreement that f shall be zero for $x = y = 0$, which

* S. Chapman, l. c., p. 379.

† For $k \geqq 1$, the theorem holds for any integrable function; see for the case $k = 1$ (the theorem holds a fortiori for $k > 1$) L. Fejér, "Untersuchungen über trigonometrische Reihen," *Math. Annalen*, vol. 58 (1904), pp. 51–69.

‡ BULLETIN, vol. 13 (1906), page 125.

shows that my statement was inaccurate, and he suggested also the hypotheses of Professor Moore which are given above.

G. A. BLISS.

REID'S THEORY OF ALGEBRAIC NUMBERS.

The Elements of the Theory of Algebraic Numbers. By LEGH WILBER REID, with an Introduction by DAVID HILBERT. New York, The Macmillan Company, 1910. xix + 454 pp.

PROFESSOR Reid's book is the result of an attempt to present the theory of algebraic numbers through a somewhat detailed study of the numbers belonging to special quadratic realms. The first four chapters, constituting one third of the whole book, are devoted to the theory of rational integers with special emphasis upon those properties of these numbers "which find their analogues in the general theory." There can be no question but that this procedure helps the reader to appreciate the character of the generalization made through the introduction of the ideal numbers, but it is difficult to see why it is necessary to use so much space for the introduction of material easily available elsewhere, when such a course makes it necessary to omit some of the more important parts of the theory for which the book was written. The presentation of this material is good, though rather lengthy.

In the proof of the unique factorization law for rational integers much emphasis is laid upon the sequence of three well-known theorems. The first of these affirms the validity of the euclidean algorism for finding the highest common factor of two integers; the second relates to the solution in integers of the equation

$$ax + by = 1,$$

where a and b are relatively prime; while the third asserts that when the product of two integers is divisible by a prime at least one of the integers is divisible by the prime. These three theorems the author refers to as Theorems A, B, and C, respectively.

Chapters V, VI, and VII are devoted to the discussion of the integers in the realms $K(i)$, $K(\sqrt{2})$, and $K(\sqrt{-3})$. The basis, the discriminant, the units of the realm are discussed

separately for each one of these realms and without reference to the general quadratic realm, though, as the author points out, many of the proofs apply directly to the general case. For each of the three realms the unique factorization law is proven exactly as in the realm of rational integers, i. e., Theorems A, B, and C with proper modifications are shown to hold. The constant repetition certainly serves to fix the main points in the reader's mind, but it grows somewhat monotonous.

In Chapter VIII the whole process is begun over again for the special realm $K(\sqrt{-5})$, and an effort is apparently made to bring the reader with as great a shock as possible to the realization of the fact that in the special realm $K(\sqrt{-5})$ the euclidean algorism can not be carried out and, consequently, the unique factorization law can not be proven by methods used for the realms previously discussed. Be that as it may, the statement of Theorem A is given for the fifth time and this time for a realm for which it is not true. This method of presentation is not without its advantages, but the book would have been less open to criticism if the author had not gone to the trouble of dignifying an untrue statement by the name of a theorem, solely for the purpose of disproving it. Indeed, since it has been known for a long time that Theorem A holds by limitation only, it is doubtful if it deserves the prominence given it by the author in a book designed to be an introduction to the general theory.

Having shown that the unique factorization law can not be proven for $K(\sqrt{-5})$ by the method employed in demonstrating it for the realms previously discussed, the author introduces a few simple examples to show that it is not true, not only for this realm, but for some other realms as well, and proceeds at once to the introduction of ideals defined for the special quadratic realm $K(\sqrt{-5})$. Several definitions, such as the definitions for an equality of ideals, principal ideals, multiplication of ideals, unit and prime ideals, are given only to be repeated in almost identical words in a later chapter. However much we may be inclined to criticise the repetition that is to be found in these chapters on special quadratic realms, we must recognize the fact that they contain a mass of concrete illustrations which will be of great service to the beginner.

The valuable part of the book so far as American readers are concerned is to be found in the last six chapters. In these chapters the author discusses General theorems concerning algebraic numbers, the General quadratic realm, Ideals of a quadratic realm, Congruences whose moduli are ideal, Units of the general quadratic realm, and Ideal classes of a quadratic realm. The presentation is clear and is illustrated at nearly every point by numerous well-chosen numerical examples that will be a most welcome help to the beginner. Great pains are taken to point out analogies existing between the quadratic ideals and the algebraic integers.

The general excellence of this part of the book is marred by an unfortunate error on page 316 in what purports to be a proof of the theorem that the ideals of any realm for which the unique factorization holds are all principal ideals. In place of the erroneous statement, already noticed in a review by Professor Dickson,* the author has substituted the following to replace lines 5–16: "The converse of this, that, whenever the unique factorization theorem in its usual form holds for the integers of the realm, the ideals of the realm are all principal ideals, is true, but a further development of the theory of ideals is necessary for its proof. If, however, Theorem B holds for any realm, it is evident that not only does the unique factorization theorem hold but if

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_r)$$

be any ideal of the realm and δ the greatest common divisor of $\alpha_1, \alpha_2, \cdots, \alpha_r$, then, as in $K(i)$, there exist integers

$$\xi_1, \xi_2, \cdots, \xi_r,$$

such that

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \cdots + \alpha_r\xi_r = \delta.$$

Hence, we have

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_r) = (\alpha_1, \alpha_2, \cdots, \alpha_r, \delta) = \delta,$$

a principal ideal."

This correction has been embodied in a two page list of "Additional Errata" printed for distribution since the first copies of the book were sent out.

It may seem like mere quibbling to say that the author fails to place in the clearest possible light the exact meaning of the

* *Science*, n. s., vol. 33, p. 188.

phrase "the restoration of the unique factorization law." On page 254 we read "When this factorization has been performed we find that every integer of $K(\sqrt{-5})$ can be represented in one and only one way as the product of prime ideal numbers." After making this direct statement, it is scarcely fair to the beginner for the author to relegate to a footnote the important statement: "We speak of the factorization of an integer α into its ideal factors, meaning thereby always the factorization of the principal ideal defined by α."

The chief defect of the book appears to be an over-elaboration of the material which is intended to be introductory to the main purpose. This elaboration of elementary material so increases the bulk of the book that any mention of the applications of the theory of quadratic ideals, or any development of the theory for higher realms is impossible. Some account of the relation between quadratic ideals and quadratic forms, and a brief account of the geometric interpretation of ideals would have added greatly to the value of the book. Moreover, to justify the title, the author was in duty bound to outline at least the simpler parts of the theory for the general case.

On the other hand, the latter half of the book is the easiest available approach for our American students to the elements of one of the most beautiful of modern mathematical creations. Furthermore, the author has put in a very clear light the historical sequence of the ideas which led to the development of the theory.

A few minor errors in proof reading, not given in the list of "Additional Errata" mentioned above, have been discovered by the reviewer. The book is well printed and contains an excellent index. A particularly valuable feature is the large number of numerical examples, many of them worked out, which are scattered throughout the text.

It is perhaps in order to mention an earlier review of this book which appeared in *Science* for February 3, 1911. In this review a distinguished mathematician, who has not even given the author's name correctly, has criticized the book severely in point of both accuracy and form of presentation. The reviewer has pointed out with clearness and force the weak points of the book, but has made some statements which, coming as they do from one of the ablest mathematicians of this country, ought not to be allowed to pass unnoticed.

For example, the reviewer commenting upon the author's method of showing that Theorem A does not hold for the realm $K(\sqrt{-5})$, says "after stating Theorem A and devoting fifteen lines to its proof, the author informs us that the theorem therefore fails." The failure of Theorem A, or its equivalent, is a fact of prime importance in bringing to light the simplest example of an "incomplete holoidal realm," the recognition of which was the starting point for the introduction of the whole theory of ideals. Moreover, the context makes it perfectly clear, even before going to the proof of the fact, that the theorem fails for $K(\sqrt{-5})$. Concerning the method employed differences may well exist, as I have already intimated, but the quotation given above puts the author in a wholly erroneous light.

In speaking of topics omitted from the book the same reviewer includes the class number. If he had gone over the table of contents carefully, he would have found that more than half of the last chapter, containing twenty-five pages, is given up to the class number.

Again, the reviewer, deploring the omission of references, says: "But to give hundreds of references to a certain report (excellent though it may be) and to completely ignore the literature and not even mention the names of the discoverers of the theorems, is against all scientific traditions." There are in the book one hundred and fifty-eight foot-note references to authors. Of these thirty-eight, not "hundreds," refer to Hilbert's Report. Frequent mention is made of the work of Fermat, Gauss, Jacobi, Dirichlet, Dedekind, Kummer, Kronecker, Minkowski, and the references are usually exact. We find, indeed, no reference to the books of Hensel or of König, and it is perhaps unfortunate that Sommer's excellent "Vorlesungen über Zahlentheorie," to which several references are made, is omitted from the half page list of books and reports given in the preface. But taking it all in all, the references are about as numerous as one should expect in a strictly elementary book.

To the present writer it seems that criticism based in part, at least, upon careless and inaccurate statements, and freely interspersed with exclamation points, is, to use the former reviewer's own phrase, "against all scientific traditions."

E. B. SKINNER.

HEINRICH MARTIN WEBER.

Festschrift Heinrich Weber zu seinem siebzigsten Geburtstag am 5 März 1912. Gewidmet von Freunden und Schülern. Mit dem Bildnis von H. WEBER in Heliogravüre und Figuren im Text. Leipzig und Berlin, B. G. Teubner, 1912. viii + 500 pages. Price 24 marks.

IT will not have been without a sharp pang that former pupils read of the death, at Strassburg on May 17, of Heinrich Martin* Weber. The breadth of his mathematical knowledge, the fascinating delivery of his lectures, the great modesty of his bearing, the charity and rich sympathy of his nature in association with young men, the charming hospitality of his home (at least before the death of Mrs. Weber a dozen years ago)—are all recalled by those so fortunate as to have been his pupils.

Professor Weber was the eldest son of the noted historian Georg Weber and was born at Heidelberg, March 5, 1842. He was educated at the Lyceum in Heidelberg and at the universities of Heidelberg, Leipzig, and Königsberg. His doctor's thesis (Heidelberg, 1866) dealt with a contribution to the "Theory of singular solutions of partial differential equations of the first order." In 1866 he became privatdocent, and in 1869, ausserordentlicher professor of mathematics in the University of Heidelberg. He married in 1870. In 1873 he was appointed ordentlicher professor in the University of Königsberg; in 1883 professor in the Polytechnische Hochschule, Berlin; in 1884 professor at the University of Marburg; in 1892 professor at the University of Göttingen; and finally in 1895 professor in Strassburg, where his only sister, wife of Heinrich Holtzmann, professor of theology in the University, was living. Here he remained till his death.

Professor Weber's scientific publications have been numerous, the papers appearing for the most part in *Crelle* and in the *Mathematische Annalen*. His first book, on the Theory of Abelian Functions of Deficiency Three, was published in 1876. In 1891 appeared his Elliptic Functions and Algebraic Numbers, afterwards incorporated into his classic Lehrbuch

* Cf. *Allgemeine Deutsche Biographie*, vol. 41, p. 302.

der Algebra (2 volumes, 1895–96; 2d edition, 3 volumes, 1898,* 1899,* 1908; abridgment, 1 volume, 1912; French translation of volume I, 2 volumes 1898). His article on "Complex multiplication" appeared in the Encyklopädie der mathematischen Wissenschaften in 1900. In this year and the following were published the two volumes of the so-called fourth edition of The Partial Differential Equations of Mathematical Physics based on Riemann's lectures;† this was not a new edition however, but rather an entirely new work by Weber, with the same purposes in view. The fifth edition‡ was published 1910–12. The three volumes of the remarkably popular Encyclopaedia of Elementary Mathematics§ edited by Weber and Wellstein, first published 1903–1907, have already gone through three editions and have been expanded into four volumes.

Then, too, Weber was editor of Riemann's Collected Works (1876; 2d edition 1892), of the German edition of Poincaré's "Value of Science" (1906; 2d edition 1910‖); of the first published (1911) volume of the great new edition of Euler's Works, besides being joint editor of Franz Neumann's Works (1906) and of the *Mathematische Annalen.*

Europeans have a pleasant way of taking time to celebrate anniversary occasions both within and without the family circle. Friends and former pupils of Moritz Cantor on his seventieth birthday (in 1899) presented him with a "Festschrift" of over 600 pages, containing in addition to an expression of appreciation and good wishes, some 30 original contributions to the history of mathematics by as many historians. In like manner those who held Professor Weber in high esteem prepared a similar memorial for his seventieth birthday, and this is the volume before us for review.

The heliogravure frontispiece portrait is singularly characteristic and life-like. Besides the memoirs, only the following introductory message is to be found in the volume:

"*Highly honoured Professor!* A number of your friends and

* Reviewed by J. Pierpont in this BULLETIN, vol. 4 (1898), pp. 200–234; vol. 5 (1899), pp. 480–482.

† Reviewed in this BULLETIN, vol. 8 (1901), pp. 81–85 (J. S. Ames).

‡ Reviewed in this BULLETIN, vol. 18 (1911), pp. 87–89 (O. D. Kellogg); vol. 19 (1913), pp. 482–483 (J. B. Shaw).

§ Reviewed in this BULLETIN, vol. 10 (1904), pp. 200–204 (D. E. Smith); vol. 14 (1908), pp. 499–501 (H. S. White); vol. 17 (1911), p. 546 (F. W. Owens); vol. 19 (1913), pp. 422–423 (J. B. Shaw).

‖ Reviewed in this BULLETIN, vol. 19 (1913), pp. 252–253 (J. B. Shaw).

pupils have united to present to you, on your seventieth birthday, this book as a token, of abiding remembrance and of their sincere admiration and attachment. We have believed that we could not better express our regard and feeling of gratitude than by a joint original production which in many-sided content is a reflection of the richness of your own life work. For with rare versatility you have directed your ceaseless activity in different departments of science, in algebra, number theory, theory of functions, in mechanics and mathematical physics, you have created works of permanent value, you have opened up new fields of research, while at the same time, in comprehensive works which have become the common property of all mathematicians, you have fixed in panoramic fashion the scientific possessions of our time.

"But not alone as investigator and teacher do we honour you to-day; our good wishes are likewise a token of appreciation of the personality worthy of all honour, an offering to the true ever trusty friend, to the lofty-minded man, who even under severest trial has preserved his true goodness of heart and belief in ultimate triumph for high ideals.

"The firm of B. G. Teubner by the printing and publication of this work also wish to express their sincere esteem, and we conclude with heartiest good wishes."

The memoirs are arranged in alphabetical order according to authors:

J. Bauschinger (of Strassburg), "On the Laplace compared with the Gauss method for determining an orbit" (pages 1–10); O. Blumenthal (Aachen), "Remarks on the singularities of analytic functions of several variables" (11–22); R. Dedekind (Braunschweig), "On the Zeller proof of the quadratic reciprocity theorem" (23–36); A. Eichenwald (Moscow), "On the field of light waves in reflection and refraction" (37–56); P. Epstein (Strassburg), "The generalizations of Kronecker's boundary formula" (57–74); R. Gans (Strassburg), "Is gravitation of electromagnetic origin?" (75–94); H. Hahn (Czernowitz), "General proof of Osgood's theorem of the calculus of variations for simple integrals" (95–110); L. Henneberg (Darmstadt), "On the center of gravity of funicular nets and on special space reciprocal figures of graphical statics" (111–129); D. Hilbert (Göttingen), "On the notion of class of differential equations" (130–146); E. V. Huntington (Cambridge, Mass.), "A new

approach to the theory of relativity"* (147–169); A. Kneser (Breslau), "Remarks on the number of extremes of curvature of closed curves and on related questions in a non-euclidean geometry" (170–180); A. Krazer (Karlsruhe), "On the theory of multiple sums of Gauss" (181–197); A. Loewy (Freiburg), "On homomorphic groups and the influence by adjunctions on the rationality groups of linear homogeneous differential equations" (198–227); L. Mandelstam (Strassburg), "On an application of integral equations in the theory of optical images" (228–241); L. Maurer (Tübingen), "On transformation relations" (242–251); R. v. Mises (Strassburg), "Contribution to the oscillation problem" (252–282); T. Reye (Strassburg), "On the pencil congruence (2, 2) of Hirst" (283–290); F. Schur (Strassburg), "On the generation of surfaces of the second degree by correlative sheaves" (291–297); M. Simon (Strassburg), "Cusanus as mathematician" (298–337); A. Sommerfeld (Munich), "On propagation of light in dispersing media" (338–374); A. Speiser (Strassburg), "On the composition of binary quadratic forms" (375–395); P. Stäckel (Karlsruhe), "Periodic functions and systems of infinitely many equations" (396–409); E. Study (Bonn), "Groups of two-sided collineations" (410–413); H. E. Timmerding (Braunschweig), "On the molecular theory foundation for the theory of elasticity" (414–421); W. Voigt (Göttingen), "The electrostatic field in a stationary light-stream" (422–427); P. Volkmann (Königsberg), "Historical and critical studies on the notion of causation"† (428–442); R. H. Weber (Rostock), "On the proof of uniformity in the theory of the conduction of heat" (443–456); J Wellstein (Strassburg), "Algebraic uniformization of algebraic functions" (457–479); C. Wirtz (Strassburg), "On the figure of the moon" (480–500).

R. C. ARCHIBALD.

BROWN UNIVERSITY,
June 7, 1913.

* Read before the American Mathematical Society, October 28, 1911. Also published in *Philosophical Magazine*, ser. 6, vol. 23, No. 4, pp. 494–513, April, 1912.

† Professor Weber delivered a Prorektor oration at the University of Königsberg entitled, "On Causation in Natural Science" (publ. at Leipzig, 1881). To the works mentioned above as edited by Weber should be added five volumes of Ostwald's Klassiker series: Rosenhain, 1895; Göpel, 1895; Jacobi, 1895; Lagrange, 1898, Gauss, 1903.

SHORTER NOTICES.

Elements of Plane Trigonometry. By D. A. MURRAY. New York, Longmans, Green, and Company, 1911. ix + 136 pp. + 95 pp. of tables.

PROFESSOR MURRAY's Elements gives a much briefer treatment of the subject than the Plane Trigonometry previously published by him. Among the topics receiving more or less special emphasis are the variation, periodicity, and graphs of functions, general expressions for all angles having one function in common, and methods of checking solutions.

Problems requiring the use of trigonometric tables are introduced almost at the very beginning of the book. One wonders if it might not be better to defer the first use of such tables, which, for some reason or other, the student usually finds rather difficult, until he shall have grown more familiar with the meaning of trigonometric functions by use of the functions of 45°, 30°, and 60°, and by the solution of problems such as those involving the determination of the remaining functions of an angle from a single given function, and the solution of right triangles from one side and a function of one angle.

In the matter of oblique triangles, the various formulas connected with the solution of such triangles are first worked out merely as "relations between the sides and angles of a triangle." Then, in a separate chapter the solution of triangles is taken up and the formulas previously developed are made use of. While this arrangement may serve to emphasize "theory for the sake of theory," it would seem that a combination of the two chapters in one might, without detracting much, if any, from this emphasis, make the work more simple and interesting for the student.

CORA B. HENNEL.

A Brief Course in Analytic Geometry. By J. H. TANNER and JOSEPH ALLEN. New York, American Book Company, 1911. x + 282 + xxiv pp.

THE student beginning the study of analytic geometry usually finds himself, almost at the very start, in the midst of

a mass of formulas, without realizing what they are all about. In Tanner and Allen's Brief Course in Analytic Geometry, an abridgment of the same authors' Elementary Course, this difficulty appears to have been overcome. After a brief introductory chapter, giving a review of the parts of algebra and trigonometry the student will have use for, the notion of rectangular coordinates is carefully explained; elementary applications, including the distance between two points, the area of a triangle, slopes of parallel and perpendicular lines, are given; and the relation between the graph and the equation is clearly stated and brought home to the student by numerous examples given for him to solve. Then, and not until then, is the student introduced to the various forms of the equation of the straight line, and the numerous formulas connected with it.

In the treatment of the conic sections, also, the authors' arrangement is somewhat unusual. After the circle has been discussed in detail, the equations of the conic sections with reference to coordinate axes in any position are worked out and the general second degree equation is discussed. Then follows a treatment of secants, tangents, normals, and diameters for all conics, and finally a separate study of each conic with its geometric properties.

That the authors believe that students learn by doing is evidenced by the sixty-six lists of exercises, containing eight hundred and fifty-one problems, given in the first part of the book, which covers plane geometry. That they realize also the difficulties of literal notation and general proofs is shown by their "introduction of the demonstration of general theorems by numerical examples."

The subject of solid geometry is very briefly treated in the second part of the book, which contains also a short discussion of higher plane curves.

CORA B. HENNEL.

Gedenktagebuch für Mathematiker. Von Prof. Dr. FELIX MÜLLER. Dritte Auflage. Leipzig und Berlin, Teubner, 1912. iv + 121 pp.

MANY office and desk calendars of more or less pretentious proportions are widely distributed—and early—each year and many of these give honorable mention on the proper

dates throughout the year to some noteworthy deed or name on which the mind might like to dwell for an instant in the midst of the labors of the day.

In the lists of men whose birthdays, or the anniversaries of whose deaths are considered worthy of mention, we find many famous statesmen, authors, painters, musicians, and the like but, alas! very few if any mathematicians, physicists, or astronomers. Why should not the order of prominence be reversed? And it is in the Gedenktagebuch under review. In a neat little volume of 121 pages are gathered for each day of the year, with great care and the kindly spirit which seems to radiate from the genial face of the author whose frontispiece adorns the book, a great mass of miscellaneous information concerning the deeds and names of celebrities in the fields of mathematics, physics, and astronomy from the time where history becomes authentic to the present.

Many a man now living may not agree with the selection of names as listed for honorable mention—especially if his is not included. Such criticism will always be directed at any catalog of men famous for their deeds and jealous of their rank. Besides the Gedenktagebuch aims to be international in scope. Those who feel that others—or they—should have this mention not now given them might fill in at the proper places on the alternate blank pages any such additional facts as would please them or add to this feature of the history of the three sciences whose noteworthy deeds and dates are so carefully chronicled.

ERNEST W. PONZER.

Technische Infinitesimalrechnung. Von Prof. Dr. F. EBNER, Oberlehrer an der Königl. höheren Maschinenbauschule zu Aachen. Berlin, Verlag von Otto Salle, 1912. vii + 172 pp.

IN attempting to classify properly this interesting pamphlet the reviewer has come to the conclusion that it might serve a most excellent purpose as a correspondence school text in the calculus for engineering students. It is primarily a collection of problems involving fundamental calculus notions which arise in engineering practice. Sufficient detail is given in the solutions of these problems to enable the reader to follow readily through to the results obtained.

A summary of the contents of the book would show that the author has selected his material from many distinct engineering fields and that every application of the principles of the calculus included would be considered fundamental and discussed thoroughly in some particular engineering course in any technical school of merit.

Since the flavor of actual engineering practice pervades every one of the 172 pages we naturally expect to find, as we do, much use made of figures and diagrams to illustrate the text, and great emphasis placed on the merit of the geometric intuition in developing the theory. The differential is used frequently and approximations are numerous, but the reader is nowhere worried by the presence of involved demonstrations employing epsilons and limits.

Instructors in the calculus, if interested, will find in the pamphlet under review some new material, several new viewpoints, and many applications of principles—all of which might aid them to add new life and vigor to their courses.

ERNEST W. PONZER.

The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies. By A. G. WEBSTER. Second edition. Leipzig, B. G. Teubner, 1912. xii + 588 pp.

THE second edition of Professor Webster's Dynamics has appeared in less than ten years after the first, and the fact that it is substantially identical with the first is evidence of the value of the work. The book grew out of the lectures delivered by the author to students of physics, and presents in compact form a treatment of so much of the science of dynamics as is considered an essential part of the equipment of an investigator in physics. The classical treatises in English as well as other languages are too bulky for a student to read completely, and they presuppose frequently a knowledge of mathematics which the student does not possess. "The attempt has been made to treat what is essential to the understanding of physical phenomena, leaving out what is chiefly of mathematical interest." The wisdom of the selection of material is attested by the success of the first edition. It is assumed that the student has a good knowledge of the calculus, but not of differential equations or higher analysis. This does not mean that subjects involving advanced mathe-

matics are avoided, but that the necessary mathematical theory is developed in the text; for example, the treatment of the brachistochrone is preceded by a sketch of the fundamental notions of the calculus of variations.

The book is divided into three parts. The first part treats the general principles and the motion of a material point; the second considers the dynamics of rigid bodies, and the third is devoted to the theory of the potential and the dynamics of deformable bodies. The brevity of the work is appreciated by a comparison with the standard treatise of Appell, where each of the three parts occupies a volume equal in size to the single volume under consideration. Professor Webster's success in presenting so much material in this compact form is due to the rapidity and directness with which results are obtained. There is no wasting of words and no unnecessary repetition, and, at the same time, no ambiguity. The statements of the fundamental principles are clear and concise and the discussions of their significance are particularly illuminating. It is to be regretted that there are no problems. Plenty of problems can be found in other treatises, but if a selection had been made for this book with the same good judgment displayed in the preparation of the text it would have been appreciated by the instructor using the book in the classroom.

While the selection of topics and method of exposition have been determined by the needs of the student of physics, the book is equally valuable to the student of pure mathematics. The constant reference to actual practical use of the principles and the reproduction in the text of drawings made by physical experiments (e. g., the spherical pendulum, page 50, and the dynamical top, pages 270, 293, 294, etc.) designed to realize the results of the theory have a peculiar interest to the student of mathematics who is less familiar with such matters than the worker in the laboratory. As a reference book for the more mature investigator this volume is quite as convenient as for the student approaching the subject for the first time.

The author asks pardon for errors of proofreading, which are numerous but, fortunately, obvious. Most of them are of the kind that should be eliminated by the professional proofreader in the printing office and probably would not have occurred if the book had been published in an English-speaking country.

Professor Webster is to be congratulated on the well-deserved success of his work and we may justly take pride in this product of American scholarship.

W. R. LONGLEY.

NOTES.

THE concluding (October) number of volume 14 of the *Transactions of the American Mathematical Society* contains the following papers: "Applications and generalizations of the conception of adjoint systems," by MAXIME BÔCHER; "On a certain class of self-projective surfaces," by E. J. WILCZYNSKI; "On the representation groups of given abstract groups," by G. A. MILLER; "On the accuracy of trigonometric interpolation," by DUNHAM JACKSON; "On a simple type of irregular singular point," by G. D. BIRKHOFF; "On quadratic residues," by J. MCDONNELL; "A set of five independent postulates for Boolean algebras, with application to logical constants," by H. M. SHEFFER; "Formal modular invariants with application to binary modular covariants," by MILDRED SANDERSON.

THE December number (volume 15, number 2) of the *Annals of Mathematics* contains: "On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$" (continued), by R. D. CARMICHAEL; "Geometric characterization of isogonal trajectories on a surface," by J. LIPKA; "On the second variation, Jacobi's equation, and Jacobi's theorem for the integral $\int F(x, y, x', y')dt$," by ARNOLD DRESDEN; "Quartic surfaces invariant under periodic transformations," by F. R. SHARPE and F. M. MORGAN; "Postulate sets for abelian groups and fields," by W. A. HURWITZ.

UNDER the auspices of the International commission a congress on the teaching of mathematics will be held at Paris, April 1–5, 1914, in the halls of the Sorbonne. The chief subjects of discussion by the congress will be the introduction of the first notions of the calculus and of primitive functions in the secondary schools (lycées, Gymnasia), and the teaching of mathematics to engineering students.

THE address on "Molecular theories and mathematics," delivered by Professor EMILE BOREL at the dedication of the Rice Institute, October 10–12, 1912, has been published in the annual report of the Smithsonian Institution for 1912, pages 167–186.

FOR several years volumes 5 and 7 of the *Proceedings of the Edinburgh Mathematical Society* have been out of print, and it has been practically impossible to procure them even in second hand condition. These volumes have now been reprinted by the Society and are available for the completion of sets.

THE Italian society for the advancement of science held its annual meeting at Siena, September 22–27. The following papers were presented at the general sessions: "The principles of mechanics," by A. GARBASSO; "Thermodynamics of liquids and gases," by S. LUSSANA; "Astronomy and historical chronology," by E. MILLOSEVICH; "Phosphorescence and fluorescence; statements and theories," by A. POCHETTINO; "The actuary and actuarial science," by G. TOIA. The following papers were presented before the section of pure and applied mathematics: "On the flow of water through conduits," by L. CONTI; "On the sufficient conditions in the calculus of variations," by F. ENRIQUES; "The southern and eastern deviations of falling bodies," by G. GIANFRANCESCHI; "Volterra's distorsions in solids of revolution," by E. LAURA; "On Torricelli's theorem," by T. LEVI-CIVITA.

UNIVERSITY OF PARIS.—The following courses in mathematics are announced for the present semester, beginning on November 3: By Professor G. DARBOUX: General principles of differential geometry and their applications to partial differential equations, two hours.—By Professor E. GOURSAT: Operations of the calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Uniform monogenic functions of one complex variable, one hour.—By Professors P. PAINLÉVÉ and C. GUICHARD: Statics, general theorems of dynamics, elements of analytic mechanics, movement of fluids, two hours.—By Professor C. GUICHARD and Drs. H. LEBESGUE and E. MONTEL: General mathematics, two hours.—By Professor J. BOUSINNESQ: Interior friction of fluids with application to continuous flow and gradual extinction

of waves, two hours.—By Professor G. KOENIGS: Thermodynamic and experimental study of thermic motors, general principles of dynamics of machines, two hours.—By Dr. L. CAHEN: Elements of the modern theory of numbers, Fermat's last theorem, two hours. Conferences will be held by Professor GUICHARD and Messrs. DRACH, MONTEL, SERVANT, and VESSIOT. Courses in the Ecole Normale are given by Professors BOREL and CARTAN and Messrs. LEBESGUE and VESSIOT.

PROFESSOR MAXIME BÔCHER, exchange professor from Harvard University, will give a two-hour course on: The work of Sturm and Liouville in differential equations; its modern developments.

DR. M. LAPALLY has been appointed docent in mathematics at the technical school of Munich.

THE Italian national academy of sciences (the XL) has awarded its gold medal to Professor MAX ABRAHAM, of the technical school at Milan, for his researches on gravitational fields.

AT the University of Washington, Mr. ALLEN F. CARPENTER has been promoted to an assistant professorship. Dr. L. E. WEAR, Dr. L. L. SMAIL, and Dr. N. ALTSHILLER have been made instructors in mathematics. Professor G. I. GAVETT has been granted leave of absence for the year.

AT the Iowa State College the following changes have been made in the mathematical department. Miss M. ROBERTS has been promoted to the rank of professor; Miss J. COLPITTS has been promoted to an associate professorship: Dr. W. M. JONES, Mr. G. A. CHANEY, and Mr. G. W. SNEDECOR have been appointed assistant professors; Miss G. HERR has been appointed instructor in mathematics. Professor E. A. PATTENGILL has returned from his leave of absence and has resumed his regular work.

MR. C. G. SIMPSON, of Columbia University, has been appointed assistant professor of mathematics in Pennsylvania State College.

DR. H. M. SHEFFER, recently instructor in mathematics in Cornell University, has been appointed instructor in philosophy in the University of Minnesota.

PROFESSOR A. W. WHITNEY, of the University of California, has resigned to accept a position in the State board of insurance.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APPELL (P.). Eléments d'analyse mathématique. 3e édition entièrement refondue. Paris, Gauthier-Villars, 1913. 8vo. 8+700 pp. Fr. 24.00

AUTONNE (L.). Notice sur les recherches mathématiques de Léon Autonne. Paris, Gauthier-Villars, 1913. 4to. 36 pp.

BEUTEL (E.). Die Quadratur des Kreises. (Mathematische Bibliotek, Band XII.) Leipzig, Teubner, 1913. 8vo. 75 pp. M. 0.80

BLISS (G. A.). See LECTURES.

BURKHARDT (H.). Theory of functions of a complex variable. Authorized translation from the 4th German edition with the addition of figures and exercises, by S. E. Rasor. Boston, Heath, 1913. 8vo. 450 pp. Cloth. $4.00

DINGELDEY (F.). Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. 2ter Teil: Integralrechnung. (Teubner's Sammlung, Band XXXII, 2.) Leipzig, Teubner, 1913. 8vo. 2+382 pp. M. 13.00

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome II, volume 4, fascicule 1: E. von Weber-Floquet, Propriétés générales des systèmes d'équations aux dérivées partielles. Equations linéaires du premier ordre.—E. von Weber-Goursat, Equations non-linéaires du premier ordre. Leipzig, Teubner, 1913. 8vo. Pp. 1–160. M. 6.00

FEHR (H.). Compte rendu du Congrès de Cambridge. (Publications du comité central, 2e série, 1re fascicule.) Genève, Georg, 1912. 8vo. 97 pp. Fr. 2.50

GALLATLY (W.). The modern geometry of the triangle. 2d edition. London, Hodgson, 1913. 8vo. 8+126 pp.

GEVREY (M.). Sur les équations aux derivées partielles du type parabolique. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 211 pp.

HENSEL (K.). Zahlentheorie. Berlin, Göschen, 1913. 8vo. 12+356 pp. Cloth. M. 10.80

HOENIKA (L. W. DE). Nouvelle démonstration de la théorie des droites parallèles d'Euclide. (Texte russe et texte française.) Moscow. 8vo. 100 pp.

KASNER (E.). See LECTURES.

KEMPE (A.). Der grosse Fermatsche Satz. Versuch einer Beweisführung. 2te, verbesserte Auflage. Amsterdam, Versluys, 1913. 22 pp.

KNOBLAUCH (J.). Grundlagen der Differentialgeometrie. Leipzig, Teubner, 1913. 8vo. 10+634 pp. Cloth. M. 20.00

KÖNIGSBERGER (L.). Die Mathematik eine Geistes- oder Naturwissenschaft? (Festrede.) Heidelberg, Winter, 1913.

KUNTZE (F.). Denkmittel der Mathematik im Dienst der exakten Darstellung erkenntnisstheoretischer Probleme. (Philosophische Vorträge, veröffentlicht von der Kantgesellschaft. Nr. 3.) Berlin, Reuther & Reichard, 1913. M. 1.00

LECTURES on mathematics, the Princeton colloquium. I: Fundamental existence theorems, by G. A. Bliss. II: Differential-geometric aspects of dynamics, by E. Kasner. New York, American Mathematical Society, 1913. 8vo. 230 pp. $1.50

MITZSCHERLING (A.). Das Problem der Kreisteilung. Ein Beitrag zur Geschichte seiner Entwicklung. Mit einem Vorwort von H. Liebmann. Leipzig, Teubner, 1913. 8vo. 170 pp. M. 7.00

NIELSEN (N.). Matematiken i Danmark 1528–1800. Bidrag til en bibliografisk-historisk oversigt. Kjöbenhavn, 1912. Kr. 7.00

PASCH (M.). Lecciones de geometria moderna. Traducción anotada de la primera edición alemana por J. G. A. Ude y J. R. Pastor. Madrid, Arias, 1913. 8vo. 11+288 pp.

PROCEEDINGS of the Fifth International Congress of Mathematicians (Cambridge, August 22–28, 1912). Edited by E. W. Hobson and A. E. H. Love. Volume I: Part 1, Report of the congress; Part 2, Lectures, communications to section I. 500 pp.—Volume II: Communications to sections II–IV. 657 pp. Cambridge, University Press, 1913. 30s.

RASOR (S. E.). See BURKHARDT (H.).

RIABOUCHINSKY (D.). La fonction $|z|$. Essai d'un calcul des valeurs absolues. Moscow, Kouchnirоff. 4to. 28 pp.

ROUGIER (L.). Henri Poincaré et la mort des vérités nécessaires. (*La Phalange.*) Paris, Crès, 1913. 8vo. 22 pp.

RUTGERS (K. W.). Over de voortbrenging v. oppervlakken v. d. 6. graad door projectieve oppervlakkenbundels. Groningen, 1912.

SCHMIDT (L.). Ueber die zu einem gegebenen linearen Komplex gehörigen kubischen Raumkurven (Kubische Nullkurven eines Nullraums). (Diss.) Strassburg, 1913.

SER (J.). Essai de linéométrie. 1re partie. Paris, Gauthier-Villars, 1913. 8vo. 83 pp. Fr. 2.75

SMART (E. H.). A first course in projective geometry. London, Macmillan, 1913. 8vo. 298 pp. 7s. 6d.

SPITZ (G.). Zur Theorie der Elemente höherer Ordnung in der Ebene und im Raume. (Diss.) Greifswald, 1912. 8vo. 54 pp.

TREU (J.). Rotations- und Schraubenflächen konstanter positiver Total-Krümmung sowie solche von konstanter mittlerer Krümmung. (Diss.) Halle, 1913. 8vo. 32 pp.

VALLE (R. DELLA). Le funzioni con argomento complesso estese sulle superficie sferiche di Riemann. S. Maria C. V., Cavotta, 1913. 8vo. 31 pp.

VOSS (F.). Die Klassifikation der Kurven zweiter Ordnung und zweiter Klasse bei Plücker. (Diss.) Rostock, 1912. 8vo. 111 pp.

WARGNY (C.). Historia de las matematicas. Santiago, Cervantes. 8vo. 375 pp. S. 6.00

WILSON (H.). Untersuchung einer linear-quadratischen Berührungstransformation. (Diss.) Rostock, 1913. 8vo. 59 pp.

WOOD (P. W.). The twisted cubic, with some account of the metrical properties of the cubical hyperbola. Cambridge, University Press, 1913. 8vo. 99 pp. 2s. 6d.

II. ELEMENTARY MATHEMATICS.

AMMERMAN (C.). See FORD (W. B.).

BAUDOIN (P.). See BOURLET (C.).

BERICHTE UND MITTEILUNGEN, veranlasst durch die Internationale Mathematische Unterrichtskommission. VIII: P. Stackel, Nachruf auf Peter Treutlein; W. Lietzmann, Der Kongress in Cambridge. M. 1.60. IX: H. Dressler, Mathematische Lehrmittelsammlungen. M. 1.00. Leipzig, Teubner, 1913. 8vo.

BOURLET (C.). Eléments d'algèbre. 1er et 2e cycles. 9e édition revisée. Paris, Hachette, 1913. 16mo. 324 pp. Cartonné. Fr. 2.00

BOURLET (C.) et BAUDOIN (P.). Leçons de géométrie et éléments de géométrie descriptive. Paris, Hachette, 1913. 16mo. 5+432 pp. Cartonné. Fr. 4.00

BOUVART (C.) et RATINET (A.). Règles et formules usuelles servant de supplémentales tables de logarithmes. 4e édition. Paris, Hachette, 1913. 8vo. 47 pp. Fr. 0.60

BOWLEY (A. L.). A general course of pure mathematics from indices to solid analytical geometry. London, Clarendon Press, 1913. 8vo. 284 pp. 7s. 6d.

BRENKE (W. C.). See LONG (E.).

BURALI-FORTI (C.) e RAMORINO (A.). Elementi di algebra. 4a edizione. Torino, Gallizio, 1914. 8vo. 7+142 pp. L. 1.75

CHATELAIN (E.). L'enseignement de la géométrie. Une réforme dans ses rapports avec nos circonstances locales. La Chaux-de-Fonds, Imprimerie coopérative, 1913. 8vo. 24 pp.

COURS de trigonométrie rectiligne. Par F. G. M. Paris, Gigord, 1913. 8vo. 250 pp.

CRACKNELL (A. G.). The school algebra; with answers. Matriculation edition. London, Clive, 1913. 8vo. 8+420+65 pp. 4s. 6d.

DUMONT (E.). Cours d'arithmétique théorique et pratique, suivi d'une note sur les théories logiques des nombres. Bruxelles, DeBoeck. 8vo. 16+501 pp. Fr. 6.00

ELÉMENTS de géométrie comprenant des notions sur les courbes usuelles. Par F. J. Paris, Gigord, 1913. 16mo. 12+523 pp.

EMMERICH (A.). Leitfaden und Uebungsbuch der Stereometrie. Weinheim, Ackermann, 1913.* M. 2.20

FORD (W. B.) and AMMERMAN (C.). Plane geometry. Edited by E. R. Hedrick. New York, Macmillan, 1913. 12mo. 9+213+31 pp. Cloth. $0.80

FREISE (W.). Behandlung der Reihen im Unterricht. Göttingen, Huth. 8vo. 108 pp.

HACCOUR (M.). Introduction au cours d'algèbre. 3e édition, augmentée. Namur, Wesmael-Charlier, 1912. 75 pp. Cartonné. Fr. 0.85

HAMELIN (G.). See NAUD (L.).

LIETZMANN (W.), GECK (E.), und CRAMER (H.). Neue Erlasse in Bayern, Württemberg und Baden. (Internationale Mathematische Unterrichtskommission. Band II, Heft 8.) Leipzig, Teubner, 1913. 8vo. 6+49 pp. M. 1.50

LONG (E.) and BRENKE (W. C.). Algebra, first course. New York, Century Co., 1913. 12mo. 8+283 pp. Cloth. $1.10

MANUEL d'algèbre et de trigonométrie. Par une réunion de professeurs. Paris, Gigord, 1913. 16mo. 205 pp.

MANUEL de géométrie. Par F. G. M. Paris, Gigord, 1913. 16mo. 16+591 pp.

NAUD (L.) et HAMELIN (G.). Cours résumé de mathématiques. Nouvelle édition. Paris, Courrier des examens. 8vo. 224 pp. Fr. 2.50

PORINIOT (L.). Cours pratique d'algèbre appliquée. 4e édition. Namur, Wesmael-Charlier, 1912. 126 pp. Cartonné. Fr. 1.50

——. Leçons de géométrie pratique. Tamines, Duculot-Roulin, 1912. 96 pp. Cartonné. Fr. 1.50

RAMOLINO (A.). See BURALI-FORTI (C.).

RATINET (A.). See BOUVART (C.).

REGAL (R.). Application de l'algèbre à la résolution des problèmes d'arithmétique. Marseilles, Laurent. 8vo. 103 pp.

SAINTE-LAGUÉ (A.). Introduction au cours de mathématiques. Paris, Dunod et Pinat, 1913. 8vo. 8+512 pp. Fr. 7.00

SERRET (G. A.). Elementi di trigonometria. 14a edizione. Firenze, Le Monnier, 1913. 16mo. 127 pp. L. 1.50

——. Trattato di trigonometria. Traduzione con modificazioni e aggiunte di G. Tolomei. Nuova edizione, completamente rifatta. Firenze, Le Monnier, 1913. 8vo. 271 pp. L. 3.00

SOMERVILLE (F. H.). Elementary algebra; revised. New York, American Book Co., 1913. 12mo. 44^7 pp. Cloth. $1.00

UMLAUF (K.). Mathematik und Naturwissenschaften an den deutschen Lehrerbildungsanstalten. Leipzig, Teubner, 1912. 8vo. M. 3.60

VASNIER (M.). Cours de géométrie. 3e partie: Courbes et surfaces usuelles. Paris, 1913. 8vo. 167 pp.

VIELLEFOND (A.). Précis de géométrie. Paris, Hachette, 1913. 16mo. 2+521 pp. Fr. 3.50

III. APPLIED MATHEMATICS.

BOMSEL. Cours de mécanique générale. Paris, Millet, 1913. 8vo. 267 pp.

BURALI-FORTI (C.) et MARCOLONGO (R.). Analyse vectorielle générale. Volume II: Applications à la mécanique et à la physique. Pavia, Mattei, 1913. 8vo. 12+144 pp. L. 5.00

CAPELLE (H.). Die mathematische Geographie und ihre Nutzanwendung. Berlin, Mittler, 1913. M. 8.00

FASSBINDER. Sur la dynamique des systèmes variables et la rotation de la terre. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 57 pp.

FRICK (P.). Mécanique et électricité industrielles, I. Paris, Berger-Levrault, 1913. 8vo. 296 pp. Fr. 5.00

HINDRICHS (O.). Tabellen zur Berechnung von Tagen, Zinszahlen und Zinsen. In deutscher, englisher, französischer, italienischer, russischer und spanischer Sprache. Essen, Literatur-Verlag. 8vo. 21+188 pp. M. 20.00

HÖFLER (A.). Didaktik der Himmelskunde und der astronomischen Geographie. Leipzig, Teubner, 1913. 8vo. 12+414 pp. M. 12.00

JACKSON (D. C.). Alternating currents and alternating current machinery. New edition, rewritten and enlarged. New York, Macmillan, 1913. 8vo. 954 pp. Cloth.

JACOBY (H.). Astronomy; a popular handbook. New York, Macmillan, 1913. 8vo. 13+435 pp. Cloth. $2.50

MARCOLONGO (R.). See BURALI-FORTI (C.).

MAYCOCK (W. P.). Electric circuit theory and calculations. London, Whittaker, 1913. 370 pp. 3s. 6d.

MAYER (G. D.). Etudes dynamiques des moteurs à cylindres rotatifs. Traduit de l'italien par O. Pomilio. Paris, Dunod et Pinat, 1913. 8vo. 12+126 pp. Fr. 4.50

M'LACHLAN (N. W.). Practical mathematics. London, Longmans, 1913. 8vo. 192 pp. 2s. 6d.

MORITZ (F.). Les moteurs thermiques dans leurs rapports avec la thermodynamique. Paris, Gauthier-Villars, 1913. 8vo. 6+298 pp. Fr. 13.00

NERNST (W.). Experimental and theoretical applications of thermodynamics to chemistry. New Haven, Conn., Yale University Press. 12mo. 123 pp. Cloth. $1.25

OSORIO (A.). Théorie mathématique de l'échange. Traduit par J. d'Almada. Paris, Giard et Brière, 1913. 8vo. 18+395 pp. Fr. 10.00

OUDOT (E.). Cours élémentaire de machines marines. Paris, Dunod et Pinat, 1913. 8vo. 6+204 pp. Fr. 4.50

PERRIN (J.). Les atomes. (Collection E. Borel.) Paris, Alcan. 16mo. 16+296 pp. Fr. 3.50

PERRY (J.). Drehkreisel. Deutsch von A. Walzel. 2te verbesserte und erweiterte Auflage. Leipzig, Teubner, 1913. 8vo. M. 2.40

PETROVITCH (S.). Course in rational mechanics. (Russian.) Volume I: Kinematics. 4+11+312 pp. Volume II: Dynamics of a particle. 5+263 pp. Saint-Petersburg, 1912. 8vo.

POINCARÉ (H.). La dynamique de l'électron. Paris, Dumas, 1913. 8vo. 64 pp.

POLSTER (H.). Kinematik. Leipzig, 1912. M. 0.80

POMILIO (O.). See MAYER (G. D.).

PRIDEAUX (E. B. R.). Problems in physical chemistry with practical applications. New York, Van Nostrand. 8vo. 323 pp. $2.00

ROCHE (L.). Sur la surface des ondes dans la polarisation rotatoire magnétique et dans quelques phénomènes plus generaux. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 99 pp.

SILBERSTEIN (L.). Vectorial mechanics. London, Macmillan, 1913. 8vo. 206 pp. 7s. 6d.

SOUTHALL (J. P. C.). Geometrical optics. 2d edition. New York, Macmillan, 1913. 8vo. 648 pp. Half leather. $5.50

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